A MULTICOMMODITY MODEL OF FUTURES PRICES: USING FUTURES PRICES OF ONE COMMODITY TO ESTIMATE THE STOCHASTIC PROCESS OF ANOTHER

GONZALO CORTAZAR*
CARLOS MILLA
FELIPE SEVERINO

This article proposes a multicommodity model of futures prices of more than one commodity that allows the use of long-maturity futures prices available for one commodity to estimate futures prices for the other. The model considers that commodity prices have common and commodity-specific factors. A procedure for

The authors thank the researchers of the FINlabUC, Laboratorio de Investigación Avanzada en Finanzas, Pontificia Universidad Católica de Chile, for helpful discussions and excellent research assistance. They also acknowledge the financial support of FONDECYT (Grant 1070771), FONDEF (Grant D03I1039), and Fundacion COPEC-Universidad Católica (Grant PC00021).

*Correspondence author, Departamento de Ingeniería Industrial y de Sistemas, Escuela de Ingeniería, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago, Chile. Tel: 562-354-4272, Fax: 562-552-1608, e-mail: gcortaza@ing.puc.cl

Received August 2007; Accepted January 2008

■ *Gonzalo Cortazar is a Professor at the Departamento de Ingeniería Industrial y de Sistemas, Escuela de Ingeniería, Pontificia Universidad Católica de Chile in Santiago, Chile.*

■ *Carlos Milla and Felipe Severino are researchers at the FINlabUC, Laboratorio de Investigación Avanzada en Finanzas, Pontificia Universidad Católica de Chile in Santiago, Chile.*

The Journal of Futures Markets, Vol. 28, No. 6, 537–560 (2008)

Published online in Wiley InterScience (www.interscience.wiley.com).
DOI: 10.1002/fut.20322

choosing the number of both types of unobservable-Gaussian factors is presented. Also, it is shown how commodities with and without seasonality may be jointly modeled and how to estimate the model using Kalman filter. Results for the West Texas Intermediate–Brent and for the West Texas Intermediate–unleaded gasoline models presented show strong improvements over the traditional individual-commodity models, with much lower out-of-sample errors and better volatility estimates, even when using fewer factors. Jrl Fut Mark 28:537–560, 2008

INTRODUCTION

There is an evolving literature on how to model the stochastic behavior of commodity futures prices. The relevance of these models is at least two-fold. First they are used to estimate prices for contracts for which there are no market prices, and second they provide an estimation of the volatility term structure, which is required to value option-like derivatives or to estimate risk exposures.

Commodity models have been evolving in several dimensions. First, the number of risk factors has been increasing from early one-factor models to two-, three-, and four-factor models, with considerable gain in flexibility and adjustment to different term-structure behavior (Brennan & Schwartz, 1985; Cortazar & Naranjo, 2006; Cortazar & Schwartz, 1994; Gibson & Schwartz, 1990).

A second aspect in which models have been evolving is in the way how the drift and the factors are modeled. Initial models considered simple geometric Brownian motions, whereas more advanced models include alternative factor-dynamics specifications including seasonality, time varying risk premiums and mean reversion (Casassus & Collin-Dufresne, 2005; Dai & Singleton, 2000; Laughton & Jacoby, 1993; Manoliu & Tompaidis, 2002; Schwartz, 1997; Sørensen, 2002).

A third dimension of model evolution has been on the estimation procedures, including simple cross-section model calibration, traditional Kalman filtering with complete data panels, and extended Kalman procedures with time-dependent number of daily observations that requires no data aggregation and makes a better use of available data (Cortazar & Naranjo, 2006; Cortazar & Schwartz, 2003; Sørensen, 2002).

A fourth dimension in which models differ is in the volatility specification. Most models consider a constant volatility, whereas some propose a Unspanned Stochastic Volatility (USV) specification (Trolle & Schwartz, 2006) that seems to better fit volatility structures at the expense of some loss on term-structure fitting.

In this study a new dimension for model evolution that considers extending individual-commodity models into a multicommodity setting is proposed. The basic intuition is that for commodities with correlated returns, price variations on contracts for one commodity should be useful information for the other. This allows the use of information on the behavior of long-maturity futures prices available for one commodity to estimate futures prices for another commodity that only has short-maturity contracts.[1] Another advantage of using these multicommodity models arises when estimating the spread between two commodities is of interest. By jointly modeling both commodities more stable spreads should be obtained compared with the alternative of individually modeling both commodities.

The proposed multicommodity model considers that commodity prices have a set of common factors that explain their correlation, in addition to some commodity-specific factors. A procedure for choosing the number of common and commodity-specific unobservable-Gaussian factors is presented. Also, it is shown how commodities with and without seasonality may be jointly modeled and how to estimate the multicommodity model using a Kalman filter.

Two empirical implementations of the multicommodity model are presented: First a model for the West Texas Intermediate (WTI) and the Brent oil contracts, with the former commodity having much longer-maturity contracts than the latter, and second a model for the WTI and the unleaded gasoline in which the second commodity not only has shorter maturities, but also a strong seasonality.

Results for both model implementations show strong improvements over the traditional individual-commodity models, with much lower out-of-sample errors and better volatility estimates, even when using fewer factors.

The study is organized as follows. The next section presents the general multicommodity model, with and without seasonality. The third section shows how to estimate the model using the Kalman filter. The fourth section explains a model selection procedure for choosing the number of common and of commodity-specific factors. The fifth section shows the results of implementing the model for two pairs of commodities: WTI–Brent and WTI–unleaded gasoline. The last section concludes.

THE MULTICOMMODITY MODEL

Model Definition

The proposed multicommodity model is based on the canonical representation of Dai and Singleton (2000) for interest rates and represents an extension of Cortazar and Naranjo (2006) for more than one commodity.

First, an m-commodity $(p, k_1, \ldots, k_m)$ model is described, in which all commodities share p common factors and have k_i commodity-i specific factors. Later the model is extended to include seasonality.

[1]As pointed out by the authors' referee, the problem of forecasting or hedging non-traded securities arises because markets are incomplete. See Henderson (2002) or DeTemple and Sundaresan (1999) for models on the valuation of non-traded assets.

Let Y_i be defined as follows:

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_m \end{bmatrix}_t = \log \begin{bmatrix} S_1 \\ \vdots \\ S_i \\ \vdots \\ S_m \end{bmatrix}, i = 1, \ldots, m \text{ commodities} \tag{1}$$

with

$$Y_{it} = \tilde{h}_i' X_{it} + c_{it}$$

where $\tilde{h}_i$ is a vector of dimension $n \times 1$, c_{it} is a time-dependent function, and X_{it} is a vector of n state variables that follows a multivariate Orstein–Uhlembeck stochastic process.

Then,

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_m \end{bmatrix}_t = \begin{bmatrix} \tilde{h}_1 \\ \vdots \\ \tilde{h}_i \\ \vdots \\ \tilde{h}_m \end{bmatrix} X_t + C_t = \mathbf{h} \cdot X_t + C_t \tag{2}$$

where $\mathbf{h}$ is an $m \times n$ matrix, X_t is a state-variable vector $n \times 1$, and C_t is a time-dependent matrix $m \times 1$.

The dynamics of the state variables is defined as

$$dX_t = (-\tilde{\mathbf{A}} X_t + \tilde{\mathbf{b}})dt + \tilde{\mathbf{\Sigma}} d\tilde{\mathbf{w}}_t \tag{3}$$

where $\tilde{\mathbf{A}}$ is an $n \times n$ semi-positive matrix, $\tilde{\mathbf{b}}$ is a constant vector, $\tilde{\mathbf{\Sigma}}$ is an $n \times n$ matrix, and $d\tilde{\mathbf{w}}_t$ is a vector of n correlated Brownian motion increments such that

$$(d\tilde{\mathbf{w}}_t)(d\tilde{\mathbf{w}}_t)' = \tilde{\mathbf{\Theta}} dt. \tag{4}$$

Assuming that X_t follows a non-stationary process and applying a linear transformation

$$\mathbf{T}(X_t) = \boldsymbol{\varphi} + \mathbf{L} X_t \tag{5}$$

the canonical multicommodity model is defined as

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_i \\ \vdots \\ Y_m \end{bmatrix}_t = \begin{bmatrix} \log S_1 \\ \vdots \\ \log S_i \\ \vdots \\ \log S_m \end{bmatrix}_t =$$

$$\begin{bmatrix} \overbrace{1 & \cdots & 1}^{p} & \overbrace{1 & \cdots & 1}^{k_1} & \cdots & \overbrace{0 & \cdots & 0}^{k_i} & \cdots & \overbrace{0 & \cdots & 0}^{k_m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \delta_{i1} & \cdots & \delta_{ip} & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \delta_{m1} & \cdots & \delta_{mp} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{bmatrix} X_t + \begin{bmatrix} 1 \\ \vdots \\ \delta_{i1} \\ \vdots \\ \delta_{m1} \end{bmatrix} \mu_t \quad (6)$$

where p is the number of common state variables, k_i is the number of specific factors for commodity i, $n = p + \sum_{i=1}^{m} k_i$ is the total number of state variables, X_t is an $n \times 1$ vector of state variables, μ is the long-term growth rate, and δ_{ij} is the weight of state variable j for commodity i.

The dynamics of the vector of state variables X_t is

$$dX_t = (-KX_t)dt + \Sigma d\mathbf{w}_t \quad (7)$$

with

$$K = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n \end{pmatrix}, \; \Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}$$

$$\boldsymbol{\Theta} dt = d\mathbf{w}_t d\mathbf{w}_t^{'}, \; \boldsymbol{\Theta} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}. \quad (8)$$

This is a non-stationary model for the log spot price. By assuming a constant risk premium λ, the risk-adjusted process for the vector of state variables is

$$dX_t = -(\lambda + KX_t)dt + \Sigma d\mathbf{w}_t^{*} \quad (9)$$

where λ is an $n \times 1$ vector of real constants.

Futures Prices With and Without Seasonality

One of the good properties of this model is that it has an analytic expression for the futures prices. Following Cox, Ingersoll, and Ross (1981), the price at time t of a futures contract with maturity T must be equal to the expected spot price for T under the risk-neutral measure:

$$F(S_t, t, T) = E_t^Q(S_T) \tag{10}$$

and

$$\begin{aligned} F(X_t, t, T)_i &\\ = \exp\Bigg(& \delta_{i1}X_1(t) + \sum_{j=2}^{n} \delta_{ij} e^{-\kappa_j(T-t)} X_j(t) + \delta_{i1}\mu t \\ & + \left(\delta_{i1}\mu - \delta_{i1}\lambda_1 + \frac{1}{2}\delta_{i1}^2\sigma_1^2\right)(T-t) - \sum_{j=2}^{n} \delta_{ij} \frac{1-e^{-\kappa_j(T-t)}}{\kappa_j}\lambda_j \\ & + \frac{1}{2}\sum_{j\cdot l\neq 1} \delta_{ij}\delta_{il}\sigma_j\sigma_l\rho_{jl} \frac{1-e^{-(\kappa_j+\kappa_l)(T-t)}}{\kappa_j+\kappa_l}\Bigg). \end{aligned} \tag{11}$$

There are several ways to include seasonality in the model. Following Manoliu and Tompaidis (2002) the authors add a deterministic function $P(T)$ to the futures expression, thus:

$$\begin{aligned} F(X_t, t, T)_i = P(T)\exp\Bigg(& \delta_{i1}X_1(t) + \sum_{j=2}^{n} \delta_{ij} e^{-\kappa_j(T-t)} X_j(t) \\ & + \delta_{i1}\mu t + \left(\delta_{i1}\mu - \delta_{i1}\lambda_1 + \frac{1}{2}\delta_{i1}^2\sigma_1^2\right)(T-t) \\ & - \sum_{j=2}^{n} \delta_{ij} \frac{1-e^{-\kappa_j(T-t)}}{\kappa_j}\lambda_j + \frac{1}{2}\sum_{j\cdot l\neq 1} \delta_{ij}\delta_{il}\sigma_j\sigma_l\rho_{jl} \frac{1-e^{-(\kappa_j+\kappa_l)(T-t)}}{\kappa_j+\kappa_l}\Bigg) \end{aligned} \tag{12}$$

with $P(T)$ being a periodic step function, such that if T belongs to month m, then

$$\begin{aligned} P(T) &= S_m \\ \prod_{m=1}^{12} s_m &= 1. \end{aligned} \tag{13}$$

In any case, the volatility of futures returns is

$$\sigma_{F_i}^2(t, T) = \delta_{i1}^2\sigma_1^2 + \sum_{j,l\neq 1} \delta_{ij}\delta_{il}\sigma_j\sigma_l\rho_{jl} e^{-(\kappa_j+\kappa_l)(T-t)}. \tag{14}$$

MULTICOMMODITY ESTIMATION USING THE KALMAN FILTER

Kalman Filter and Incomplete Data

To adequately benefit from using the information on the prices of one commodity to calibrate the stochastic behavior of the prices of another, a joint estimation of both processes should be performed. In this section it is shown how the Kalman filter can be used to estimate a model, even if data are incomplete and for some days there are no prices (Cortazar, Schwartz, & Naranjo, 2007).

Let the following measurement equation relate a vector of observable variables $\mathbf{z}_t$ with a vector of state variables, $\mathbf{x}_t$:

$$\mathbf{z}_t = \mathbf{H}_t\mathbf{x}_t + \mathbf{d}_t + \mathbf{v}_t,\ \mathbf{v}_t \sim N(\mathbf{0}, \mathbf{R}_t) \tag{15}$$

where $\mathbf{H}_t$ is a $u_t \times n$ matrix, $\mathbf{d}_t$ is a $u_t \times 1$ vector, and $\mathbf{v}_t$ is a $u_t \times 1$ vector of uncorrelated Gaussian disturbances with mean 0 and covariance matrix $\mathbf{R}_t$.

The transition equation describes the stochastic process followed by the state variables:

$$\mathbf{x}_t = \mathbf{A}_t\mathbf{x}_{t-\Delta t} + \mathbf{c}_t + \mathbf{w}_t,\ \mathbf{w}_t \sim N(\mathbf{0}, \mathbf{Q}_t) \tag{16}$$

where $\mathbf{A}_t$ is $n \times n$, $\mathbf{c}_t$ is an $n \times 1$ vector, and $\mathbf{w}_t$ is a vector of uncorrelated Gaussian disturbances. The variance–covariance matrix of the estimation error, $\mathbf{P}_t$, is

$$\mathbf{P}_t = \mathbf{E}_t(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)'. \tag{17}$$

So for a given $\hat{\mathbf{x}}_{t-1}$ and $\mathbf{P}_{t-1}$ the estimated state variables and variance–covariance error estimation matrix for t will be

$$\hat{\mathbf{x}}_{t/t-1} = \mathbf{A}_t\hat{\mathbf{x}}_{t-1} + \mathbf{c}_t. \tag{18}$$

The following one-step-ahead prediction of the observed variables can be obtained:

$$\hat{\mathbf{z}}_{t|t-1} = \mathbf{H}_t\hat{\mathbf{x}}_{t|t-1} + \mathbf{d}_t. \tag{19}$$

When there is new information $\mathbf{z}_t$, a new estimation can be obtained:

$$\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t-1} + \mathbf{P}_{t|t-1}\mathbf{H}'_t\mathbf{F}_t^{-1}(\mathbf{z}_t - \hat{\mathbf{z}}_{t|t-1}) \tag{20}$$

$$\mathbf{P}_t = \mathbf{P}_{t|t-1} + \mathbf{P}_{t|t-1}\mathbf{H}'_t\mathbf{F}_t^{-1}\mathbf{H}_t\mathbf{P}_{t|t-1} \tag{21}$$

with

$$\mathbf{F}_t = \mathbf{H}_t\mathbf{P}_{t|t-1}\mathbf{H}'_t + \mathbf{R}_t \tag{22}$$

As Cortazar et al. (2007) pointed out, the above optimal estimates can be obtained even if the number of observations varies with time. The Kalman filter allows the dimension u_t of vectors $\mathbf{z}_t$, $\mathbf{d}_t$, $\mathbf{v}_t$ and of matrices $\mathbf{H}_t$ and $\mathbf{R}_t$ to be a function of time. This way of handling the missing observation problem may be increasingly important in multicommodity settings where not for all days and maturities contracts are traded.

Model parameter estimations, $\hat{\boldsymbol{\psi}}$, are then obtained maximizing the log-likelihood function of innovations:

$$\log L(\psi) = -\frac{1}{2}\sum_t \log\left|\mathbf{F}_t\right| - \frac{1}{2}\sum_t (\mathbf{z}_t - \mathbf{z}_t^-)'\mathbf{F}_t^{-1}(\mathbf{z}_t - \mathbf{z}_t^-). \tag{23}$$

Kalman Filter of the Multicommodity Model

In this section it is shown how to define a state-space representation of the multicommodity model.

First all commodities from 1 to m are stacked in the following way:

$$\mathbf{z}_t = \begin{pmatrix} \log F_{1,t}^1 \\ \vdots \\ \log F_{u_1,t}^1 \\ \vdots \\ \log F_{1,t}^i \\ \vdots \\ \log F_{u_i,t}^i \\ \vdots \\ \log F_{1,t}^m \\ \vdots \\ \log F_{u_m,t}^m \end{pmatrix} \tag{24}$$

with

$$u_t = \sum_{i=1}^{m} u_{i,t}$$

where $F_{u_i,t}^i$ is the price of a futures contract of commodity i at time t with maturity corresponding to the u_i, t position.

$$
\mathbf{H}_t = \begin{array}{c}
\overbrace{\hphantom{1 \quad e^{-\kappa_2\tau^1_{1,t}} \quad \cdots \quad e^{-\kappa_p\tau^1_{1,t}}}}^{p}\;
\overbrace{\hphantom{e^{-\kappa_{p+1}\tau^1_{1\,t}} \quad \cdots \quad e^{-\kappa_{p+k_1}\tau^1_{1,t}}}}^{k_1}\;\cdots\;
\overbrace{\hphantom{e^{-\kappa_{p+\sum_{j<i}k_j+1}\tau^i_{1,t}} \quad \cdots \quad e^{-\kappa_{p+\sum_{j=1}^{i}k_j}\tau^i_{1,t}}}}^{k_i}\;\cdots\;
\overbrace{\hphantom{e^{-\kappa_{p+\sum_{j<m}k_m+1}\tau^m_{1,t}} \quad \cdots \quad e^{-\kappa_{p+\sum_{j=1}^{m}k_j}\tau^m_{1,t}}}}^{k_m}
\\
\begin{bmatrix}
1 & e^{-\kappa_2\tau^1_{1,t}} & \cdots & e^{-\kappa_p\tau^1_{1,t}} & e^{-\kappa_{p+1}\tau^1_{1\,t}} & \cdots & e^{-\kappa_{p+k_1}\tau^1_{1,t}} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
1 & e^{-\kappa_2\tau^1_{u1,t}} & \cdots & e^{-\kappa_p\tau^1_{u1,t}} & e^{-\kappa_{p+1}\tau^1_{u1,t}} & \cdots & e^{-\kappa_{p+k_1}\tau^1_{u1,t}} & \cdots & 0 & \cdots & 0 & \cdots & 0 & \vdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\delta_{i1} & \delta_{i2}e^{-\kappa_2\tau^i_{1,t}} & \cdots & \delta_{ip}e^{-\kappa_p\tau^i_{1,t}} & 0 & \cdots & 0 & \cdots & e^{-\kappa_{p+\sum\limits_{j<i}k_j+1}\tau^i_{1,t}} & \cdots & e^{-\kappa_{p+\sum\limits_{j=1}^{i}k_j}\tau^i_{1,t}} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\delta_{i1} & \delta_{i2}e^{-\kappa_2\tau^i_{ui,t}} & \cdots & \delta_{ip}e^{-\kappa_p\tau^i_{ui,t}} & 0 & \cdots & 0 & \cdots & e^{-\kappa_{p+\sum\limits_{j<i}k_j+1}\tau^i_{ui,t}} & \cdots & e^{-\kappa_{p+\sum\limits_{j=1}^{i}k_j}\tau^i_{ui,t}} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\delta_{m1} & \delta_{m2}e^{-\kappa_2\tau^m_{1t}} & \cdots & \delta_{mp}e^{-\kappa_p\tau^m_{1t}} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & e^{-\kappa_{p+\sum\limits_{j<m}k_m+1}\tau^m_{1,t}} & \cdots & e^{-\kappa_{p+\sum\limits_{j=1}^{m}k_j}\tau^m_{1,t}} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\delta_{m1} & \delta_{m2}e^{-\kappa_2\tau^m_{u_m,t}} & \cdots & \delta_{mp}e^{-\kappa_p\tau^m_{u_m,t}} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & e^{-\kappa_{p+\sum\limits_{j<m}k_m+1}\tau^m_{u_m,t}} & \cdots & e^{-\kappa_{p+\sum\limits_{j=1}^{m}k_j}\tau^m_{u_m,t}}
\end{bmatrix}
\end{array}
\tag{25}
$$

Matrix $\mathbf{H}_t$ can be defined as in (25) and

$$\mathbf{d}_t = \begin{pmatrix} \nu_{1,t}^1 \\ \vdots \\ \nu_{u_1,t}^1 \\ \vdots \\ \nu_{1,t}^i \\ \vdots \\ \nu_{u_i,t}^i \\ \vdots \\ \nu_{1,t}^m \\ \vdots \\ \nu_{u_m,t}^m \end{pmatrix} \tag{26}$$

where for each commodity i

$$\nu_{j,t}^i = \delta_{i1}\mu t + \left(\delta_{i1}\mu - \delta_{i1}\lambda_1 + \frac{1}{2}\,\delta_{i1}^2\sigma_1^2\right)\tau_{j,t}$$
$$- \sum_{l=2}^{n} \delta_{il}\frac{1 - e^{-\kappa_l \tau_{j,t}}}{\kappa_l}\lambda_l + \frac{1}{2}\sum_{(l,q)\neq(1,1)}^{n} \delta_{il}\delta_{iq}\sigma_l\sigma_q\rho_{lq}\frac{1 - e^{-(\kappa_l+\kappa_q)\tau_{l,j}}}{\kappa_l + \kappa_q}. \tag{27}$$

With this state-space representation, the multicommodity model may be estimated using the Kalman filter.

THE MODEL SELECTION PROCEDURE

The Basic Idea

Previous sections have shown how to define and estimate a given $(p, k_1, \ldots, k_m)$ model for m commodities that shares p common factors and has k_i commodity-i specific factors. In this section a procedure is proposed for choosing p and k_i, without resorting to estimating all possible models and comparing their performance, which may be very expensive in computation time and not practical for real-world applications.

In individual-commodity models it has long been recognized that using principal component methods is useful to select the number of factors that should be used to explain a given data variance percentage. This was the case in Litterman and Scheinkman (1991) for bonds and in Cortazar and Schwartz (1994) for commodity futures. Also, these individual principal components have been analyzed to study the behavior of correlated commodities (Tolmasky & Hindanov, 2002).

Multicommodity models have the added difficulty of having to relate different variance–covariance matrices, to find common and commodity-specific factors. A similar problem can be found in the study of the evolution of biological species with the goal of finding their common components (Flury, 1988; Krzanowski, 1979; Phillips & Arnold, 1999).

To perform the comparison between biological species represented by their variance–covariance matrices of dimension $r \times r$, two procedures have been proposed: common principal components (CPCs), in which all components are common to the different matrices, and partial common principal components (CPC(p)), in which only p of the components are common.

Principal components are based on an eigenvalue–eigenvector decomposition of variance–covariance matrices. The first procedure (CPC) searches for a set of principal components that simultaneously explain the variance of all matrices. In this case the eigenvectors are equal for all matrices, the eigenvalues being specific for each one. The second procedure (CPC(p)) extends this approach allowing p eigenvectors to be identical in all groups while the remaining $r - p$ eigenvectors are specific for each group.

The model selection procedure proposed considers aggregating all futures data into a fixed number (r) of maturity classes. Second, estimating different factor representations (CPC(p)), computing the likelihood, eigenvalues, and eigenvectors for each representation. Finally, in order to penalize representations with a high number of parameters, the model is chosen using the Schwarz information criteria (SIC).

The benefits of this model selection procedure are that it is computationally efficient and multiple model representations can be easily explored, before engaging on the much more demanding Kalman filtering of the chosen model.

A More Detailed Description

The CPC approach assumes a level of similarity among m positive-definite covariance matrices $\Psi_1, \ldots, \Psi_m$ of dimension $r \times r$.

Then, m covariances matrices have CPCs if

$$\Psi_i = \beta \Lambda_i \beta', \; i = 1, \ldots, m \tag{28}$$

where β is an orthogonal $r \times r$ matrix and

$$\Lambda_i = \mathrm{diag}(\lambda_{i1}, \ldots, \lambda_{ir}). \tag{29}$$

The number of parameters is $r(r-1)/2$ for the orthogonal matrix β plus $m \cdot r$ for the diagonal matrices Λ_i.

It is assumed that all CPCs are well defined; that is, that for each $j \in \{1, \ldots, r\}$ there is at least one population i in which the characteristic root λ_{ij} is distinct.

The joint likelihood function of $\Psi_1, \ldots, \Psi_m$ given the sample covariance matrices $S_1, \ldots, S_m$, with $S_i \sim W_r(n_i, \Psi_i/n_i)$, is

$$L(\Psi_1, \ldots, \Psi_m) = C \times \prod_{i=1}^{m} e\mathrm{tr}\left(-\frac{n_i}{2}\Psi_i^{-1}S_i\right)(\det\Psi_i)^{-n_i/2} \tag{30}$$

where the factor C does not depend on the parameters. Maximizing the expression in (30), CPCs are estimated depending on the similarity of the different matrices (Flury, 1988).

However, if the two "species" are not actually the same, even though they do have some common factors, the CPC model may still be rejected. The CPC(p) model takes care of this problem by allowing p components to be identical for all m matrices while the remaining $r - p$ components are specific to each one. Formally, the hypothesis of CPC(p) (of order p) is

$$\Psi_i = \beta^{(i)}\Lambda_i\beta^{(i)\prime}, \; i = 1, \ldots, m \tag{31}$$

where

$$\Lambda_i = \mathrm{diag}(\lambda_{i1}, \ldots, \lambda_{ir}) \tag{32}$$

and

$$\beta^{(i)} = (\beta_c, \beta_s^{(i)}). \tag{33}$$

All $\beta^{(i)}$ are orthogonal $r \times r$ matrices. Then β_c, of dimension $r \times p$, is common to all groups, whereas $\beta_s^{(i)}$, of dimension $r \times (r-p)$, is specific. By orthogonality CPC($r-1$) implies CPC(r), which is the ordinary CPC model. The authors therefore restrict p to the range $1 \leq p \leq r - 2$. This means that the CPC(p) model requires a dimension r of at least 3.

In this case β_c and $\beta_s^{(i)}$ will be written as

$$\beta_c = (\beta_1, \ldots, \beta_p) \tag{34}$$

and

$$\beta_s^{(i)} = (\beta_{p+1}^{(i)}, \ldots, \beta_r^{(i)}), \; i = 1, \ldots, m. \tag{35}$$

Just like in the CPC approach, the authors start with m independent sample matrices $S_i \sim W_r(n_i, \Psi_i/n_i)$. Assuming the CPC($p$) model, maximizing the likelihood is equivalent to minimizing the function

$$g(\beta_c, \beta_s^{(1)}, \ldots, \beta_s^{(m)}, \Lambda_1, \ldots, \Lambda_m)$$

$$= \sum_{i=1}^{m} n_i \left[\sum_{j=1}^{r} \log \lambda_{ij} + \sum_{j=1}^{p} \frac{\beta'_j S_i \beta_j}{\lambda_{ij}} + \sum_{j=p+1}^{r} \frac{\beta_j^{(i)'} S_i \beta_j^{(i)}}{\lambda_{ij}} \right] \tag{36}$$

subject to orthogonal constraints for all $\beta^{(i)}$:

$$\beta'_h \beta_j = \begin{Bmatrix} 0 & \text{if } h \neq j \\ 1 & \text{if } h = j \end{Bmatrix}, 1 \leq h, j \leq p$$

$$\beta_h^{(i)'} \beta_j^{(i)} = \begin{Bmatrix} 0 & \text{if } h \neq j \\ 1 & \text{if } h = j \end{Bmatrix}, p \leq h, j \leq r, i = 1, \ldots, m$$

$$\beta'_h \beta_j^{(i)} = 0, i = 1, \ldots, m, 1 \leq h \leq p < j \leq r. \tag{37}$$

This is equivalent to minimizing the function

$$G = g - \sum_{h=1}^{p} \gamma_h (\beta'_h \beta_h - 1) - 2 \sum_{1 \leq h < j \leq p} \gamma_{hj} \beta'_h \beta_j - \sum_{i=1}^{m} \left[\sum_{h=p+1}^{r} \gamma_h^{(i)} (\beta_h^{(i)'} \beta_h^{(i)} - 1) \right.$$
$$\left. - 2 \sum_{p < h < j \leq r} \gamma_{hj}^{(i)} \beta_h^{(i)'} \beta_j^{(i)} + 2 \sum_{1 \leq h \leq p < j \leq r} \delta_{hj}^{(i)} \beta_h^{(i)'} \beta_j^{(i)} \right] \tag{38}$$

where the γ_h, γ_{hj}, $\gamma_h^{(i)}$, $\gamma_{hj}^{(i)}$, and $\delta_{hj}^{(i)}$ are the $[p(p+1) + m(r-p)(r+p+1)]/2$ Lagrange multipliers.

To avoid the shortcomings of multiple testing when comparing several competing models, the authors propose using the SIC that penalizes models with many parameters.

$$SIC = -2\log L + q \log(n) \tag{39}$$

where L is the likelihood function, q is the number of parameters, and n is the sample size.

Assuming c alternative models, where $q_1 < q_2 < \ldots < q_c$ denotes the number of parameters and $L_1 \leq L_2 \leq \ldots \leq L_c$ the values of the likelihood function at the maxima, then minimizing (39) is equivalent to minimizing

$$SIC(i) = -2\log \frac{L_i}{L_c} + (q_i - q_1)\log(n)\ \forall i, n_i = n. \tag{40}$$

MODEL IMPLEMENTATION AND RESULTS FOR THE WTI–BRENT AND THE WTI–UNLEADED GASOLINE MODELS

The Data

Two empirical implementations of the proposed multicommodity model are presented. Model CLCB studies the WTI oil (CL) and the Brent oil (CB) futures contracts, whereas Model CLHU analyzes the WTI (CL) and the unleaded gasoline (HU) contracts. In both models WTI has much longer-maturity contracts than the other commodity. In addition, prices of the unleaded gasoline contract exhibit seasonality.

Each of the two models is tested using three different data sets. First, for the in-sample testing, a data set is used for parameter and state-variable estimation. Second, in what the authors call traditional out-of-sample testing a data set is used only for state-variable estimation, but without re-estimating model parameters. Finally, in the extreme out-of-sample testing model prices are compared with those of a data set that is neither used for parameter nor for state-variable estimation.

For Model CLCB the in-sample data consist of daily prices for CL and CB contracts traded at NYMEX and IntercontinentalExchange, respectively, between 2001 and 2004. The longest-maturity CL contract used is seven years, whereas the longest-maturity CB contract used is 2.5 years. The traditional out-of-sample data include the same contracts traded daily between January 2005 and December 2006. Given that new long-maturity CB contracts (over 2.5 years) were issued starting on February 2005, the authors use these data only for the extreme out-of-sample testing and compare these prices with model estimations that did not consider this information at all.

For Model CLHU the in-sample data consist of daily prices for CL and HU contracts traded at NYMEX between 2000 and 2004. The longest-maturity CL contract used is seven years, whereas the longest-maturity HU contract is one year. The traditional out-of-sample data include the same contracts traded daily between January 2005 and December 2006. Given that during 2006 the HU contracts were faced out (with the last contract maturing on December 2006), and that a new reformulated gasoline contract (RB) was introduced, prices from July to December of 2006 for this new RB contract (with maturities that extend to 2007) are used for the extreme out-of-sample testing.

Model Selection

In this section the question of how many common factors to include in each of the multicommodity models is addressed.

Contracts for each of the three commodities (CL, CB, and HU) in the in-sample data set are aggregated into six groups (Cortazar & Schwartz, 1994; Litterman & Scheinkman, 1991). Then, for each pair of commodities (CL–CB and CL–HU), CPC(p) are estimated, using different numbers of common factors (from zero to five), and the SIC number (which takes into consideration both the likelihood and the number of model parameters) is computed. The number of common factors chosen for each model will be the one that minimizes the SIC.

Table I reports that for the CLCB model, this procedure proposes three common factors, whereas Table II reports that two common factors should be used for the CLHU model. To allow for each model to have at least one commodity-specific factor and also for each commodity to have at least three factors to adequately represent its dynamics, the authors finally choose (3, 0, 1) factors for the CLCB model and (2, 1, 1) factors for the CLHU model, in which (p, k_1, k_2) defines a model with p common factors and k_i commodity-specific factors for commodity i.

Model Estimation

Using the Kalman filter procedure with incomplete data panels described earlier, two multicommodity models (CLCB and CLHU) plus their four individual-commodity models, which will be used as benchmarks, are calibrated.

The performance of Model CLCB, which is specified with (3, 0, 1) factors, will be compared with an individual CL model with three factors and an individual CB model with four factors.

TABLE I

Model Selection for the CL–CB Case

	χ^2	*Number of Parameters* (q_i)	$(q_i - q_1)$	$SIC(i)$
CPC(5)	82.67	27	0	82.67
CPC(4)	76.30	28	1	82.97
CPC(3)	*38.13*	*30*	*3*	
58.14				
CPC(2)	20.71	33	6	60.74
CPC(1)	11.08	37	10	77.80
CPC(0)		42	15	100.08
n	790			

Note. SIC, Schwarz information criteria; CPC, common principal components. The table shows the *SIC*(*p*) for different number of partial common principal components CPC(*p*). Note that CPC(5) = CPC(6) and that the minimum *SIC* (*p*) is

TABLE II

Model Selection for the CL–HU Case

	χ^2	*Number of Parameters* (q_i)	$(q_i - q_1)$	$SIC(i)$
CPC(5)	301.93	27	0	301.93
CPC(4)	277.28	28	1	284.40
CPC(3)	69.74	30	3	91.10
CPC(2)	*38.25*	*33*	*6*	
80.96				
CPC(1)	17.15	37	10	88.33
CPC(0)		42	15	106.78
n	1,234			

Note. SIC, Schwarz information criteria; CPC, common principal components. The table shows the *SIC*(*p*) for different number of partial common principal components CPC(*p*). Note that CPC(5) = CPC(6) and that the minimum *SIC*(*p*) is

The performance of Model CLHU, which is specified with (2, 1, 1) factors and 12 seasonality factors, will be compared with an individual CL model with three factors and an individual HU model with three factors and 12 seasonality factors.

Table III gives the parameter estimates for all the models.

TABLE III

Parameter Estimates for Multicommodity and Individual-Commodity Models

	CLCB	*CL*	*CB*	*CLHU*	*CL*	*HU*
κ_1	–	–	–	–	–	–
κ_2	0.384	0.318	0.460	0.414	1.081	1.000
κ_3	0.911	1.106	0.697	1.184	0.455	4.001
κ_4	0.435		6.951	1.104	–	–
σ_1	0.196	0.424	0.252	0.217	0.239	0.333
σ_2	0.178	0.688	0.885	0.122	0.662	0.398
σ_3	0.336	0.977	1.000	0.325	0.286	0.362
σ_4	0.080	–	0.121	0.355	–	–
ρ_{21}	−0.389	−0.902	−0.567	−0.408	0.424	−0.604
ρ_{31}	0.370	0.892	0.525	0.152	−0.483	0.488
ρ_{41}	−0.192	–	−0.055	0.010	–	–
ρ_{32}	−0.589	−0.948	−0.965	−0.026	−0.760	−0.603
ρ_{42}	0.180	–	0.275	0.044	–	–
ρ_{43}	−0.087	–	−0.310	0.954	–	–
λ_1	0.022	0.079	0.051	0.018	0.037	0.006
λ_2	0.046	0.054	−0.068	−0.010	0.179	0.004
λ_3	−0.003	0.137	0.213	0.064	0.074	0.007
λ_4	0.017	–	−0.179	−0.170	–	–
μ	0.002	0.001	0.000	0.000	0.007	0.003
ξ_1	0.005	0.005	0.003	0.006	0.006	0.017
ξ_2	0.008	–	–	0.022	–	–
s_1	–	–	–	0.963	–	0.969
s_2	–	–	–	0.978	–	0.974
s_3	–	–	–	1.021	–	1.021
s_4	–	–	–	1.063	–	1.056
s_5	–	–	–	1.063	–	1.051
s_6	–	–	–	1.051	–	1.042
s_7	–	–	–	1.035	–	1.030
s_8	–	–	–	1.011	–	1.010
s_9	–	–	–	0.971	–	0.975
s_{10}	–	–	–	0.955	–	0.962
s_{11}	–	–	–	0.949	–	0.958
s_{12}	–	–	–	0.951	–	0.960
δ_{11}	1.000	1.000	1.000	1.000	1.000	1.000
δ_{12}	1.000	1.000	1.000	1.000	1.000	1.000
δ_{13}	1.000	1.000	1.000	1.000	1.000	1.000
δ_{14}	–	–	1.000	–	–	–
δ_{21}	1.000	–	–	0.998	–	–
δ_{22}	1.000	–	–	0.985	–	–
δ_{23}	1.000	–	–	–	–	–
δ_{24}	1.000	–	–	1.000	–	–
log L	182,458	123,558	71,267	229,490	1,085,474	48,174

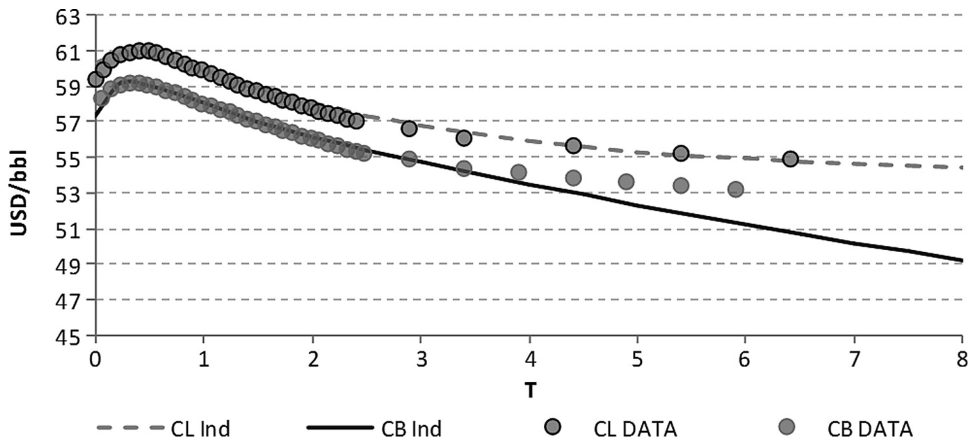

FIGURE 1
CL and CB individual-commodity models June 20, 2005. CB contracts over 2.5 years are not used for calibration.

Results

To analyze the performance of the multicommodity models, the authors compare each of them with individual-commodity models specified with the same number of factors. For example, the (3, 0, 1) CLCB model is compared with the three-factor CL model and the four-factor CB model. This is a conservative comparison in the sense that while the individual models have seven factors altogether, the multicommodity model only has four. Even with this somewhat unfair comparison the multicommodity model performs much better than the individual-commodity models, as will be shown in what follows.

There are at least[2] two measures of model performance that could be used to validate the use of a multicommodity, instead of an individual commodity, model. The first measure is *price adjustment*, or how close model prices to market transactions are, and the second measure is *volatility adjustment*, or how close model and empirical volatility term structures are.

Price Adjustments

Figure 1 shows individual model estimations for the CL and CB contracts on June 20, 2005. All CL and CB contracts with less than 2.5-year maturities are included in the traditional out-of-sample data and are very similar to model estimates. CB contracts with more than 2.5-year maturities belong to the extreme out-of-sample data set (not used for parameter or state-variable estimation) and exhibit strong differences with model estimates.

[2]A third performance measure could be *spread stability* between two commodities. The authors' preliminary results show much more stable spreads on multicommodity models than on individual models.

As can be seen in Figure 2, the multicommodity model is able to fit very well this extreme out-of-sample data set by making use of the correlation structure between both commodities to infer prices of contracts not traded.

Table IV gives the in-sample and out-of-sample mean and root mean square errors for the CL using the individual- and the multicommodity models. Table V gives the same information for the CB commodity including the extreme out-of-sample testing using contracts with maturities that exceed 2.5 years.

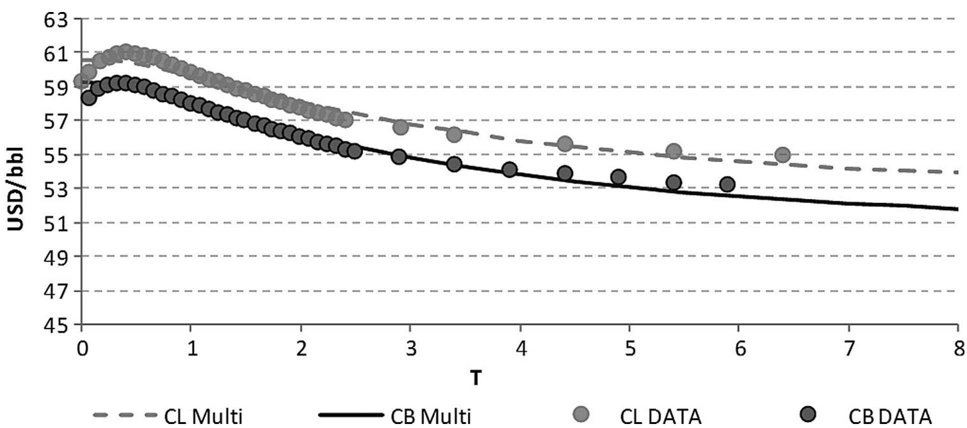

FIGURE 2
CLCB multicommodity model June 20, 2005. CB contracts over three years are not used for calibration.

TABLE IV
In-Sample and Out-of-Sample Errors for the CL Commodity

	ME CL Indiv. (%)	*ME CL Multi. (%)*	*RMSE CL Indiv. (%)*	*RMSE CL Multi. (%)*
In sample	−0.0011	−0.0012	0.48	0.54
Out sample	−0.0012	−0.0129	0.33	0.45

Note. ME, mean error; RMSE, root mean square error.

TABLE V
In-Sample and Out-of-Sample Errors for the CB Commodity

	ME CB Indiv. (%)	*ME CB Multi. (%)*	*RMSE CB Indiv. (%)*	*RMSE CB Multi. (%)*
In sample	0.0029	−0.0080	0.31	0.80
Out sample	1.1668	0.3625	2.86	1.06
Extreme out sample	4.2567	1.5560	5.15	1.74

Note. ME, mean error; RMSE, root mean square error.

Figures 3 and 4 and Tables VI and VII compare the performance of individual- and multicommodity models for the CL and HU contracts.

An analysis of the above tables and figures shows a much better behavior of multicommodity models on extreme out-of-sample testing for extrapolating

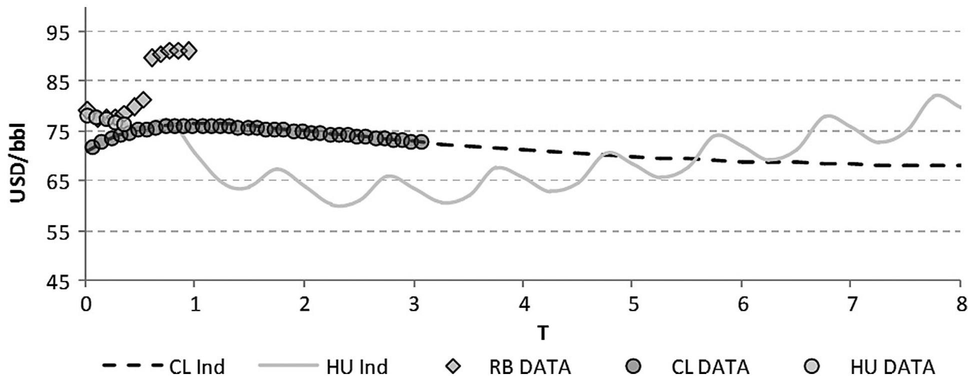

FIGURE 3
CL and HU individual-commodity models August 23, 2006. RB contracts are not used for calibration.

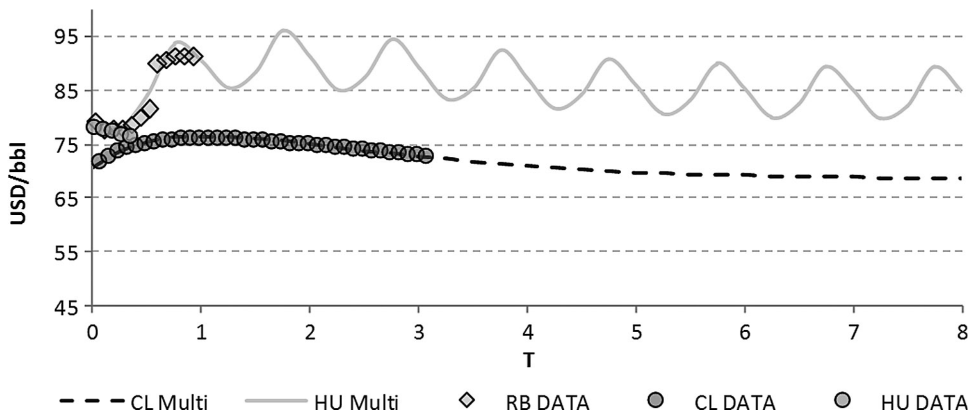

FIGURE 4
CLHU multicommodity commodity model August 23, 2006. RB contracts are not used for calibration.

TABLE VI
In-Sample and Out-of-Sample Errors for the CL Commodity

	ME CL Indiv. (%)	*ME CL Multi.* (%)	*RMSE CL Indiv.* (%)	*RMSE CL Multi.* (%)
In sample	−0.0083	−0.0038	0.54	0.54
Out sample	0.0076	0.0054	0.38	0.38

Note. ME, mean error; RMSE, root mean square error.

TABLE VII
In-Sample and Out-of-Sample Errors for the HU Commodity

	ME HU Indiv. (%)	*ME HU Multi.* (%)	*RMSE HU Indiv.* (%)	*RMSE HU Multi.* (%)
In sample	0.0082	0.1913	1.56	1.87
Out sample	0.0225	−0.6392	0.81	2.19
Extreme out sample	8.6271	−0.3812	10.38	3.73

Note. ME, mean error; RMSE, root mean square error

long-maturity prices for a commodity with only short-maturity contracts. The figures also show better spread estimates using multicommodity models. Finally, given that individual-commodity models have more independent factors, it is not surprising that their in-sample errors are smaller.

Volatility Adjustments

In this section the authors compare model volatility with empirical volatility for each commodity. Figure 5 plots the CL volatility term structure implied from the CLCB multicommodity model, the CL volatility term structure implied from the individual CL model, and empirical volatilities directly computed from the CL price transaction data.

Figure 6 plots the CB volatility term structure implied from the CLCB multicommodity model, the CB volatility term structure implied from the individual CB model, and empirical volatilities directly computed from the CB price data.

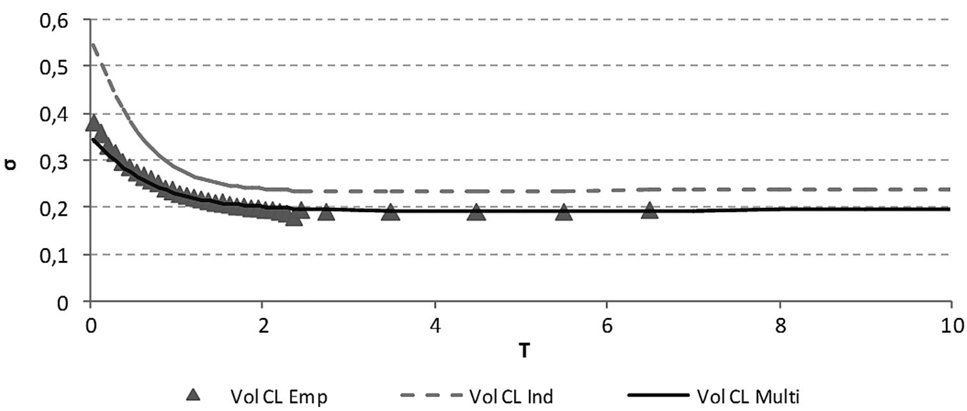

FIGURE 5
CL volatility term-structure individual- and multicommodity CLCB model.

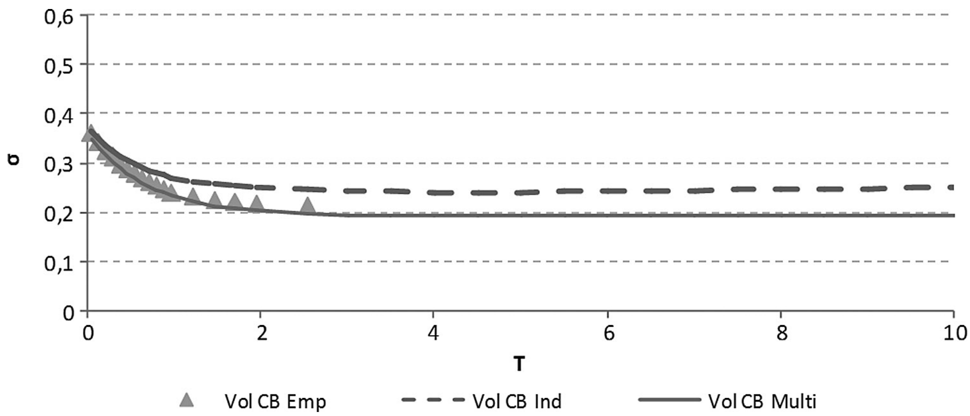

FIGURE 6
CB volatility term-structure individual- and multicommodity CLCB model.

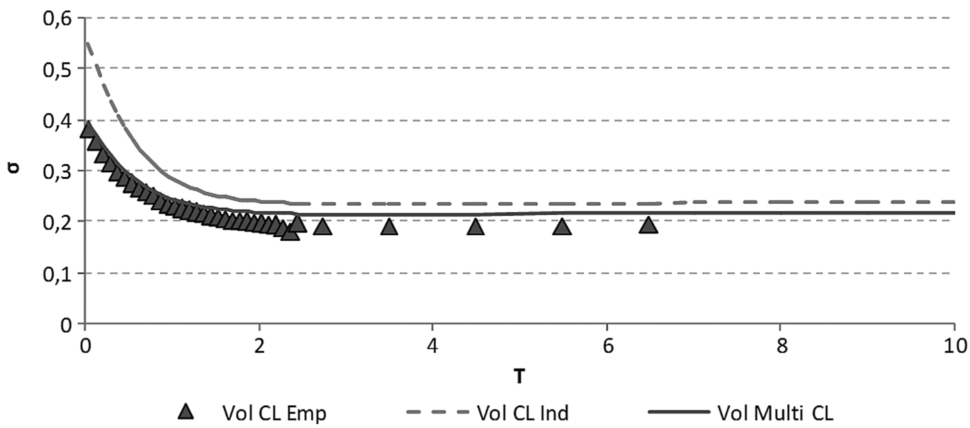

FIGURE 7
CL volatility term-structure individual- and multicommodity CLHU model.

Figure 7 plots the CL volatility term structure implied from the CLHU multicommodity model, the CL volatility term structure implied from the individual CL model, and empirical volatilities directly computed from the CL price data.

Finally, Figure 8 plots the HU volatility term structure implied from the CLHU multicommodity model, the HU volatility term structure implied from the individual HU model, and empirical volatilities directly computed from the HU price data.

It can be seen that for all the above cases, multicommodity volatility term-structure estimates track much better empirical volatilities than those implied from individual-commodity models.

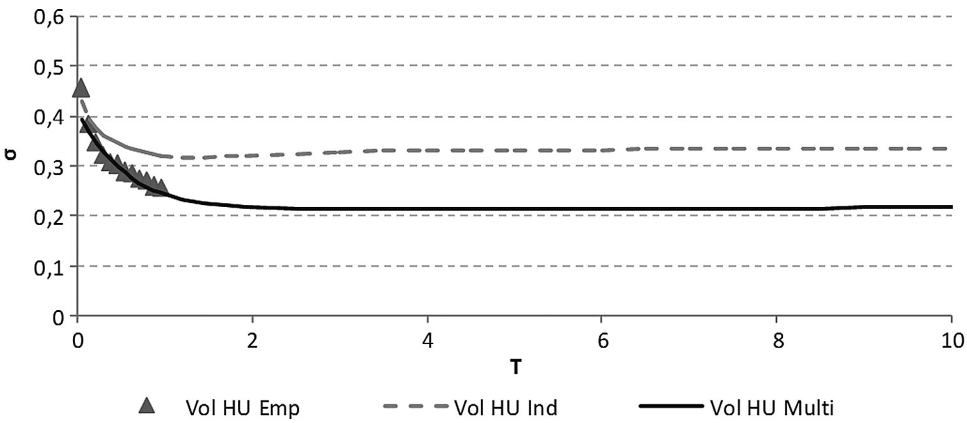

FIGURE 8
HU volatility term-structure individual- and multicommodity CLHU model.

CONCLUSION

This article proposes a multicommodity model of futures prices to explain the stochastic behavior of more than one commodity. Jointly modeling more than one commodity has the advantage of being able to use long-maturity futures prices of one commodity to estimate futures prices of another commodity that only has short-maturity contracts.

The model considers that commodity prices have a set of common factors that explain the correlation among them, in addition to some commodity-specific factors. The multicommodity model is based on the canonical representation of Dai and Singleton (2000) for interest rates and represents an extension of the individual-commodity model in Cortazar and Naranjo (2006).

A procedure for choosing the number of common and commodity-specific unobservable-Gaussian factors is presented. Also, how commodities with and without seasonality may be modeled together and how to estimate the multicommodity model using a Kalman filter are shown.

A first empirical implementation of the proposed multicommodity model is presented for the WTI (CL) and the Brent oil (CB) contracts. A (3, 0, 1) CLCB model is chosen, with three common factors and one commodity-specific factor for the CB contract.

A second model implementation for the WTI (CL) and the unleaded gasoline (HU) is discussed. A (2, 1, 1) CLHU model is chosen, with two common factors and one commodity-specific factor for each of the two commodities. In addition, the HU contract is assumed to have 12 constants to fit its seasonal behavior.

Results for both model implementations show strong improvements over the traditional individual-commodity models, with much lower out-of-sample errors and better volatility estimates, even when using fewer factors.

The advantages of using these multicommodity models are specially clear when model estimates are compared with data not used at all in model calibration, in what the authors call extreme out-of-sample testing. Also if spreads between two commodities are of interest, using multicommodity models should provide much more stable estimates.

BIBLIOGRAPHY

Brennan, M. J., & Schwartz, E. S. (1985). Evaluating natural resources investments. Journal of Business, 58, 135–157.

Casassus, J., & Collin-Dufresne, P. (2005). Stochastic convenience yield implied from commodity futures and interest rates. The Journal of Finance, 60, 2283–2331.

Cortazar, G., & Naranjo, L. F. (2006). An *N*-factor Gaussian model of oil futures. The Journal of Futures Markets, 26, 243–268.

Cortazar, G., & Schwartz, E. S. (1994). The valuation of commodity-contingent claims. Journal of Derivatives, 1, 27–35.

Cortazar, G., & Schwartz, E. S. (2003). Implementing a stochastic model for oil futures prices. Energy Economics, 25, 215–238.

Cortazar, G., Schwartz, E. S., & Naranjo, L. F. (2007). Term structure estimation in markets with infrequent trading. International Journal of Finance and Economics, 12, 353–369.

Cox, J. C., Ingersoll, J., & Ross, S. (1981). The relation between forward prices and futures prices. Journal of Financial Economics, 9, 321–346.

Dai, Q., & Singleton, K. J. (2000). Specification analysis of affine term structure models. The Journal of Finance, 55, 1943–1978.

DeTemple, J., & Sundaresan, S. (1999). Non-traded asset valuation with portfolio constraints: A binomial approach. Review of Financial Studies, 12, 835–872.

Flury, B. (1988). Common principal components and related multivariate models. New York: Wiley.

Gibson, R., & Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. The Journal of Finance, 45, 959–976.

Henderson, V. (2002). Valuation of claims on non-traded assets using utility maximization. Mathematical Finance, 12, 351–373.

Krzanowski, W. J. (1979). Between groups comparison of principal components. Journal of the American Statistical Association, 74, 703–707.

Laughton, D. G., & Jacoby, H. D. (1993). Reversion, timing options, and long-term decision-making. Financial Management, 22, 225–240.

Litterman, R., & Scheinkman, R. (1991). Common factors affecting bond returns. Journal of Fixed Income, 47, 129–1282.

Manoliu, M., & Tompaidis, S. (2002). Energy futures prices: Term structure models with Kalman filter estimation. Applied Mathematical Finance, 9, 21–43.

Phillips, P. C., & Arnold, S. J. (1999). Hierarchical comparison of genetic variance–covariance matrices I. Using the Flury hierarchy. Evolution, 53, 1506–1515.

Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. The Journal of Finance, 52, 923–973.

Sørensen, C. (2002). Modeling seasonality in agricultural commodity futures. The Journal of Futures Markets, 22, 393–426.

Tolmasky, C., & Hindanov, D. (2002). Principal components analysis for correlated curves and seasonal commodities: The case of the petroleum market. The Journal of Futures Markets, 22, 1019–1035.

Trolle, A., & Schwartz E. S. (2006). Unspanned stochastic volatility and the pricing of commodity derivatives (NBER Working Paper No. 12744).