Available online at www.sciencedirect.com
ScienceDirect
Computers & Operations Research 35 (2008) 113 – 129
computers & operations research
www.elsevier.com/locate/cor

The valuation of multidimensional American real options using the LSM simulation method

Gonzalo Cortazar*, Miguel Gravet, Jorge Urzua

Departamento de Ingeniería Industrial y de Sistemas, Escuela de Ingeniería, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago, Chile

Available online 22 March 2006

Abstract

In this paper we show how a multidimensional American real option may be solved using the LSM simulation method originally proposed by Longstaff and Schwartz [2001, The Review of the Financial Studies 14(1): 113–147] for valuing a financial option and how this method can be used in a complex setting. We extend a well-known natural resource real option model, initially solved using finite difference methods, to include a more realistic three-factor stochastic process for commodity prices, more in line with current research. Numerical results show that the procedure may be successfully used for multidimensional models, expanding the applicability of the real options approach.

Even though there has been an increasing literature on the benefits of using the contingent claim approach to value real assets, limitations on solving procedures and computing power have often forced academics and practitioners to simplify these real option models to a level in which they loose relevance for real-world decision making. Real option models present a higher challenge than their financial option counterparts because of two main reasons: First, many real options have a longer maturity which makes risk modeling critical and may force considering many risk factors, as opposed to the classic Black and Scholes approach with only one risk factor. Second, real investments many times exhibit a more complex set of interacting American options, which make them more difficult to value. In recent years new approaches for solving American options have been proposed which, coupled with an increasing availability of computing power, have been successfully applied to solving long-term financial options. In this paper we explore the applicability of one the most promising of these new methods in a multidimensional real option setting.

Keywords: Real options; Simulation; Natural resources; Valuation; Finance

1. Introduction

Even though in the last two decades there has been an increasing literature on the benefits of using the contingent claim approach to value real assets, limitations on solving procedures and computing power have often forced academics and practitioners to simplify these real option models to a level in which they loose relevance for real-world decision making.

There are two main reasons why real option models may present a higher challenge than their financial option counterparts to be solved. First, many real options have a longer maturity which makes risk modeling critical and may

* Corresponding author. Tel.: +56 2 3544272; fax: +56 2 5521608.
E-mail address: gcortaza@ing.puc.cl (G. Cortazar).

0305-0548/$ - see front matter
doi:10.1016/j.cor.2006.02.016

force the use of several risk factors, as opposed to only one, like in the classic Black and Scholes [1] stock-option model. Second, often real investments exhibit a more complex set of nested and interacting American options, which make them more difficult to value.

In the valuation of natural resource investments, for example, until only a few years ago most commodity price models considered only one risk factor and constant risk-adjusted returns. These earlier models have several undesirable implications, including that all futures returns should be perfectly correlated and exhibit the same volatility, which is not in line with empirical evidence. In recent years, however, many multifactor models of commodity prices have been proposed being much more successful than previous one-factor models in capturing the observed behavior of commodity prices like mean-reversion and a declining volatility term-structure [2,5–7].

On the other hand, the real options literature has also evolved and models increasingly take into account the different types of flexibilities available to decision makers when managing their projects. These flexibilities include the options to abandon a project, to shut down production, to delay investments, to expand capacity, to reduce costs through learning, among many others [8–11].

The introduction of multifactor price models into these real option models with many interacting flexibilities increases the difficulty of solving them, making traditional numerical approaches, like the finite difference methods, clearly inadequate. There has been, however, new research on using some sort of computer-based simulation procedures for solving American options, which coupled with an increasing availability of computing power, has been successfully applied to solving multifactor financial options. [12–18]. One of the most promising new approaches in this literature is the LSM method proposed by Longstaff and Schwartz [19] which has been tested for some financial options of limited complexity [20–22].

In this paper we explore the applicability of the LSM method in a multidimensional real option setting. We extend the Brennan and Schwartz [23] one-factor model for valuing a copper mine initially solved using finite difference methods, to include a more realistic three-factor stochastic process for commodity prices, more in line with current research. We implement the LSM method and discuss how complexity may be reduced. Numerical results show that the procedure may be successfully used for multidimensional models, notably expanding the applicability of the real options approach.

The remainder of this paper is organized as follows. Section 2 presents the problem to be solved. It describes the classic Brennan and Schwartz [23] real option model of a natural resource investment and how we extend it to include a multifactor model of commodity prices. A brief explanation on the real options approach for valuing investments is also included. Section 3 presents the proposed computer-based simulation procedure. Section 4 discusses the results of the numerical solution to the original and to the extended Brennan and Schwartz model and some implementation issues for high-dimensional models. Finally, Section 5 concludes.

2. The problem

2.1. The Real options approach to valuation

Real option valuation (ROV), can be understood as an adaptation of the theory of financial options to the valuation of investment projects. ROV recognizes that the business environment is dynamic and uncertain, and that value can be created by identifying and exercising managerial flexibility.

Options are contingent claims on the realization of a stochastic event, with ROV taking a “multi-path” view of the economy. Given the level of uncertainty, the optimal decision-path cannot be chosen at the outset. Instead, decisions must be made sequentially, hopefully with initial steps taken in the right direction, actively seeking learning opportunities, and being prepared to appropriately switch paths as events evolve.

ROV presents several improvements over traditional discount cash flow (DCF) techniques. First it includes a better assessment of the value of strategic investments and a better way of communicating the rationale behind that value. In most traditional DCF valuations, a base value is calculated. Then, this base value is “adjusted” heuristically to capture a variety of critical phenomena. Ultimately, the total estimated value may be dominated by the “adjustment” rather than the “base value.” With ROV, the entire value of the investment is rigorously captured. Conceptually, this includes the “base value” and the “option premium” obtained from actively managing the investment and appropriately exercising options.

Second, ROV provides an explicit roadmap or "optimal policy" for achieving the maximum value from a strategic investment. Most traditional investment valuations boil down to a number, and perhaps a set of assumptions underlying that number. However, the management actions required over time to realize that value are not clearly identified. With ROV, the value estimate is obtained specifically by considering these management actions. As a result, ROV indicates precisely which events are important and the necessary actions required to achieve maximum value.

There is a broad literature on ROV and how to maximize contingent claim value over all available decision strategies. Among them, Majd and Pindyck [24] include the effect of the learning curve by considering that accumulated production reduces unit costs, Trigeorgis [25] combines real options and their interactions with financial flexibility, McDonald and Siegel [26] and Majd and Pindyck [27] optimize the investment rate, and He and Pindyck [28] and Cortazar and Schwartz [29] consider two optimal control variables.

The ROV approach has been used to analyze uncertainty on many underlying assets, including exchange rates [30], costs [31] and commodities [32]. Real asset models have included natural resource investments, environmental, new technology adoption, and strategic options, among others [32–35].

Recently real options analysis is gradually advancing into the domain of strategic management and economic organization. Bernardo and Chowdry [11] analyze the way in which the organization learns from its investment projects. A related model is presented in [36]. They study the choice between a small and a large project, where choosing the small project allows one to re-invest later in the large project. Lambrecht and Perraudin [37] introduce incomplete information and preemption into an equilibrium model of firms facing real investment decisions. Miltersen and Schwartz [38] develop a model to analyze patent-protected R&D investment projects when there is imperfect competition in the development and marketing of the resulting product. Finally, Murto et al. [39] present a modeling framework for the analysis of investments in an oligopolic market for a homogenous commodity.

In this paper, we extend and solve the well-known Brennan and Schwartz [23] model for valuing natural resource investments. Other papers on natural resource investments include [40–45], among many others.

2.2. *The Brennan and Schwartz [23] Model*

The valuation of a copper mine in [23] laid the foundations for applying option pricing arbitrage arguments to the valuation of natural resource investments. In the model the value-maximizing policy under stochastic output prices considers the optimal timing of path-dependent, American-style options to initiate, temporarily cease or completely abandon production. We now describe the optimization problem in a general framework for valuing a switching option.

Consider the Brennan and Schwartz [23] model as a switching option with value $V_t(\mathbf{x}, j)$ and cash flows $CF_t(\mathbf{x}, j)$ at time t, which depend on a vector of N state variables, $\mathbf{x} = (x^1, \ldots, x^N)$ and the state of production j. The model considers that there are K states of production and the switching option can move from one state, j, to another, i, paying the corresponding switching cost, $C_t^{j,i}(\mathbf{x})$. This state switches can be made at any of $T+1$ stages, with $t = t_0, t_1, \ldots, t_T$.

We assume, for simplicity that the process for the state variables can be risk-adjusted and that markets are complete. Thus we can use the standard option pricing technique, which means that the switching option can be valued as the discounted expectation under the risk-neutral probability measure. At maturity, we assume the switching option has no value, thus:

$$V_T(\mathbf{x}, j) = 0; \quad j = 1, \ldots, K. \tag{1}$$

The switching option can then be solved recursively as follows. Moving backwards in time, in $t = T - \Delta t$ the value of the option is maximized among all feasible future stages:

$$V_{T-\Delta t}(\mathbf{x}, j) = \max_{i=1,\ldots,K} \left\{ CF_{T-\Delta t}(\mathbf{x}, i) - C_{T-\Delta t}^{j,i}(\mathbf{x}) \right\}; \quad j = 1, \ldots, K. \tag{2}$$

At times $t = t_0, t_1, \ldots, t_{T-2\Delta t}$ the value of the option can be computed as a function of current cash flows and the conditional expectation of the value in the following period. For example in $t_{T-2\Delta t}$:

$$V_{T-2\Delta t}(\mathbf{x}, j) = \max_{i=1,\ldots,K} \left\{ CF_{T-2\Delta t}(\mathbf{x}, i) + E_{T-2\Delta t}\left[V_{T-\Delta t}(\mathbf{x}, i) \right] \mathrm{e}^{-r\Delta t} - C_{T-2\Delta t}^{j,i}(\mathbf{x}) \right\}; \\ j = 1, \ldots, K, \tag{3}$$

where r is the risk free rate between time $t_{T-2\Delta t}$ and $t_{T-\Delta t}$. $E_{T-2\Delta t}$ [.] represents the conditional expectation at time $t_{T-2\Delta t}$ under the risk neutral probability measure. Consequently, the initial value of the switching option $V_0(\mathbf{x}, j^*)$ can be solved by this backward recursion where j^* represents the initial state.[1]

To determine the critical vector of state variables $\mathbf{x^c}$ that triggers the transition between different states of production, we must find the values that equate the conditional expectations between states of production.

In the original Brennan and Schwartz [23] the project is a contingent claim on copper price which follows a one-factor model, thus:

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z, \tag{4}$$

in which μ is the instantaneous price return, σ is the return volatility and $\mathrm{d}z$ is an increment to a standard Gauss–Wiener process.

Commodity holders are assumed to receive, in addition to the price return, a convenience yield which does not accrue to the holder of a financial instrument contingent on copper, i.e. a futures contract. This convenience yield, C, is assumed to be proportional to the spot price, thus the risk-adjusted process for commodity prices may be written as:

$$\frac{\mathrm{d}S_t}{S_t} = (r-c)\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{5}$$

with r being the risk-free interest rate.

The initial amount of copper reserves is $Q_{\max}$, and the mine produces at a constant rate of q, so there are R feasible states of reserves, where

$$R = \frac{Q_{\max}}{q\Delta t}.$$

Also the mine may be open, closed or abandoned, so there are $3R$ states of production. The cost of switching between states depends on K_1, K_2 and M, with K_1 being the cost of closing an open mine, K_2 being the cost of opening a closed mine, and M the annual cost of maintaining a closed mine. The mine is abandoned at no cost when market value reaches zero. The unit cost of production is A, thus the cash-flow, when the mine is open, is

$$CF(S_t) = q(S_t - A) - \tau,$$

where τ includes annual income and royalty tax payments. In addition there is an annual property tax amounting to a fraction λ_1 or λ_0 of market value, depending on whether the mine is open or closed. When closed, the mine has no earnings, but incurs in a maintenance annual cost of M.

2.3. Extending the Brennan and Schwartz [23] Model

Initial applications of the real options approach were made in the natural resource sector mainly because of its high irreversible investments and the well developed commodity futures markets. Even though real option models, like the one we just described, have been successful in capturing many managerial flexibilities, in general they have considered very simple specifications of the price risk process, hindering the use of this approach in real-world applications.

This simple risk specification represented the state-of-the art in commodity price modeling when this approach was developed more than two decades ago. Since then much research has been done to capture in a better way the commodity price stochastic process, but real option models have not kept pace with this research, probably in part due to the added complexity to obtain numerical solutions in a multi-factor setting.

In this section we extend the Brennan and Schwartz [23] model to include a multifactor specification for uncertainty, model which in later sections will be solved numerically.

Commodity price processes differ on how convenience yield is modeled and on the number of factors used to describe uncertainty. Early models, i.e., Brennan and Schwartz [23], assumed a constant convenience yield and a one-factor Brownian motion. Later on, mean reversion in spot prices began to be included as a response to evidence that futures

[1] Later in the paper we add to this notation the subscript ω to indicate a simulated path.

return volatility declines with maturity. One-factor mean reverting models can be found, for example, in [46–48]. With one-factor models, however, all futures returns are assumed to be perfectly correlated which is not consistent with empirical evidence.

To account for a more realistic price behavior, two-factor models, with mean reversion, were introduced. Examples are [2–4]. Later, Cortazar and Schwartz [7] proposed a three-factor model for commodity prices and estimated it using oil futures, showing that the model exhibits low estimation errors.

In this paper we calibrate the Cortazar and Schwartz [7][2] three-factor model with copper futures and use it as an extension of the Brennan and Schwartz [23] model of a copper mine.

The model has three state variables, the commodity spot price, S_t, the demeaned convenience yield, y_t, and the expected long-term spot price return, v_t. Commodity spot prices follow a geometric Brownian motion. Spot price returns have an instantaneous drift equal to the expected long-term return, v_t, minus short-term deviations from the convenience yield, y_t. Both y_t and v_t are mean reverting, the first one to zero and the second one to a long-term average, $\bar{v}$.

The authors show that the three factors allow for an increased flexibility of the model which makes it able to match both the shape of the futures price curves and also the volatility term structure, two key attributes for price model selection.

The dynamics of the state variables are:

$$\frac{\mathrm{d}S_t}{S_t} = (v_t - y_t)\,\mathrm{d}t + \sigma_1\,\mathrm{d}z_1, \tag{6}$$

$$\mathrm{d}y_t = -\kappa y_t\,\mathrm{d}t + \sigma_2\,\mathrm{d}z_2, \tag{7}$$

$$\mathrm{d}v_t = a(\bar{v} - v_t)\,\mathrm{d}t + \sigma_3\,\mathrm{d}z_3, \tag{8}$$

with

$$\mathrm{d}z_1\,\mathrm{d}z_2 = \rho_{12}\,\mathrm{d}t, \quad \mathrm{d}z_1\,\mathrm{d}z_3 = \rho_{13}\,\mathrm{d}t, \quad \mathrm{d}z_2\,\mathrm{d}z_3 = \rho_{23}\,\mathrm{d}t. \tag{9}$$

Defining λ_i as the risk premium for each of the three risk factors, the risk-adjusted processes are:

$$\frac{\mathrm{d}S_t}{S_t} = (v_t - y_t - \lambda_1)\,\mathrm{d}t + \sigma_1\,\mathrm{d}z_1^*, \tag{10}$$

$$\mathrm{d}y_t = (-\kappa y_t - \lambda_2)\,\mathrm{d}t + \sigma_2\,\mathrm{d}z_2^*, \tag{11}$$

$$\mathrm{d}v_t = (a(\bar{v} - v_t) - \lambda_3)\,\mathrm{d}t + \sigma_3\,\mathrm{d}z_3^*, \tag{12}$$

with

$$(\mathrm{d}z_1^*)(\mathrm{d}z_2^*) = \rho_{12}\,\mathrm{d}t, \quad (\mathrm{d}z_2^*)(\mathrm{d}z_3^*) = \rho_{23}\,\mathrm{d}t, \quad (\mathrm{d}z_1^*)(\mathrm{d}z_3^*) = \rho_{13}\,\mathrm{d}t. \tag{13}$$

Following the same estimation procedure used in Cortazar and Schwartz [7] for oil prices, we calibrate this model for copper using all futures traded between 1991 and 1998 at NYMEX, obtaining the parameter values shown in Table 1.

The model allows for all three state variables to be correlated, providing a greater flexibility which is in line with empirical evidence. It is interesting to note that most parameter values, including the factor correlations, exhibit a sign and magnitude similar to those reported in [7] for oil. Also, the model fits the empirical data with a mean absolute error of 0.2% and exhibits similar theoretical and empirical volatilities, as shown in Fig. 1.

Using this three-factor price model to extend the Brennan and Schwartz [23] real option model we obtain a much better model specification. With this new price process, and following the general framework described in the previous section, we have that the switching option now depends on three state variables.

[2] Cortazar and Schwartz [7] is an extension of the Schwartz [3] model for commodity prices, and shares some of its good properties like mean reversion while ensuring positive prices. Other commodity price models could have been used, including square-root processes, stationary models or general affine models [49].

Table 1
Parameter values of the Cortazar and Schwartz [7] three-factor commodity price model calibrated using all copper futures traded between 1991 and 1998 at NYMEX

Parameters	Value
λ_1	−0.032
λ_2	−0.392
λ_3	−0.193
a	1.379
κ	2.850
$\bar{v}$	−0.007
σ_1	0.257
σ_2	0.906
σ_3	0.498
ρ_{12}	0.215
ρ_{23}	0.841
ρ_{13}	−0.229

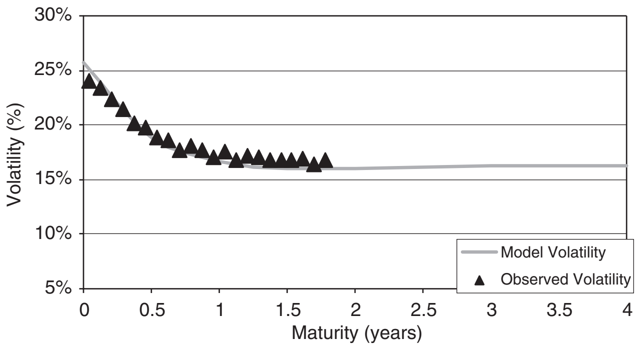

Fig. 1. Empirical and theoretical volatility term structure using the Cortazar and Schwartz [7] three-factor commodity price model calibrated using all copper futures traded between 1991 and 1998 at NYMEX.

Even though this model may be solved with traditional finite difference methods, is solved much more efficiently using the simulation method shown in the following sections.

3. Implementation

3.1. An introduction to the LSM method

We propose solving multidimensional problems, like the extended Brennan and Schwartz model, using the LSM method. To illustrate the LSM method proposed in Longstaff and Schwartz [19], we consider throughout this section a very simple copper mine that may extract all available resources instantaneously at any moment during the concession period. Also copper prices are considered in this section to follow a one-factor model. In the next section we will show how to implement the extended Brennan and Schwartz three-factor model.

Consider a simplified copper mine in which all reserves, Q, may be instantaneously extracted at any point in time incurring in a unit production cost of A. The copper spot-price, S_t, is assumed to follow a one-factor geometric Brownian motion:

$$\frac{\mathrm{d}S_t}{S_t} = (r - c)\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{14}$$

with r the risk-free interest rate and c the convenience yield.

The method starts by simulating a discretization of Eq. (14):

$$S_t = [1 + (r - c)\Delta t]\, S_{t-1} + S_{t-1}\sigma\sqrt{\Delta t}\,\varepsilon_t \tag{15}$$

with Δt the time interval in years and ε_t a random variable with a standard normal distribution.

Then, Eq. (15) is simulated through time, obtaining a price-path ω. The process is repeated N times, and a price matrix $\mathbf{S}$, with N price paths over a time horizon T, is obtained.

Like in any American option valuation procedure, the optimal exercise decision at any point in time is obtained as the maximum between the immediate exercise value and the expected continuation value. Given that the expected continuation value depends on future outcomes, the procedure must work its way backwards, starting from the end of the time horizon, T.

Starting with the last price in each path, ω, given that at expiration the expected continuation value is zero, the option value in T for the price path ω can be computed as

$$C(S_T(\omega)) = \mathrm{Max}(Q(S_T(\omega) - A); 0). \tag{16}$$

One time-step backward, at $t = T - \Delta t$, the process is repeated for each price path, but now expected continuation value must be computed. It is important to notice that at this last time-step the expected continuation value may be computed using the analytic expression for a European option.

The main contribution of the LSM method is to compute the expected continuation value for all previous time-steps by regressing the discounted future option values on a linear combination of functional forms of current state variables. Given that the way these functional forms are chosen is not straightforward, in most of the paper we use simple powers of all state variables (monomials) and their cross products which is the most common implementation of the method found in the literature. In the last section of the paper we revisit this decision and provide alternative functional forms, which in our tests have shown to be computationally efficient in multidimensional settings.

In particular, let L^j, with $j = 1, 2, \ldots M$, be the basis of functional forms of the state variable $S_{T-\Delta t}(\omega)$ used as regressors to explain the realized present value in trajectory ω, then the least square regression is equivalent to solving the following optimization problem:

$$\underset{\{\mathbf{a}\}}{\mathrm{Min}} \sum_{\omega=1}^{\mathrm{N}} \left[C(S_T(\omega))\mathrm{e}^{-r\Delta t} - \sum_{j=1}^{M} a^j L^j(S_{T-\Delta t}(\omega)) \right]^2. \tag{17}$$

The optimal coefficients $\hat{\mathbf{a}}$ are then used to estimate the expected continuation value $\hat{G}(S_{T-\Delta t}(\omega))$:

$$\hat{G}(S_{T-\Delta t}(\omega)) = \sum_{j=1}^{M} \hat{a}^j L^j(S_{T-\Delta t}(\omega)). \tag{18}$$

Fig. 2 shows discounted continuation values of our simple copper mine for all N simulated paths and the expected continuation function computed as the solution to the regression of these values on powers of the spot copper price.

Then, the optimal decision for each price path is to choose the maximum between two values: the immediate exercise and the expected continuation value.

Once we have worked ourselves backwards until $t = 0$, we have a final vector of continuation values for each price-path, which averaged provides us with an estimation of its expected value, which in turn, when compared with the immediate exercise value gives the option value at time $t = 0$:

$$\text{Option value} = \mathrm{Max}[Q(S_0 - A); \hat{G}(S_0)]. \tag{19}$$

3.2. Implementing the extended Brennan and Schwartz model

In this section we show how to implement the LSM approach to solve the Brennan and Schwartz [23] model for any price process, including the options to abandon a mine, to close an open mine and to open a closed mine.

Fig. 3 may be useful to understand the nature of the problem by describing all possible states during the simulation. It can be seen that as time evolves from 0 to T, the state variables that describe the three-factor dynamics for copper price,

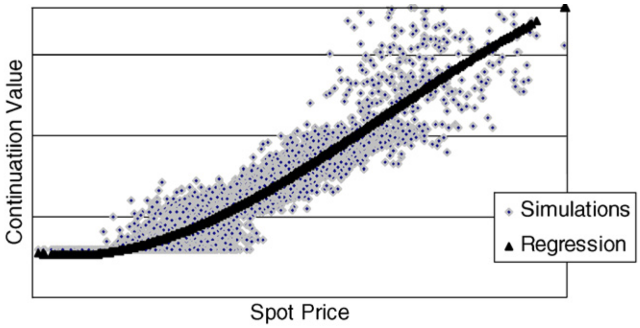

Fig. 2. Implementation of the LSM in the simple copper mine: discounted continuation values for all N simulated paths and expected continuation function computed from a regression on powers of the spot copper price.

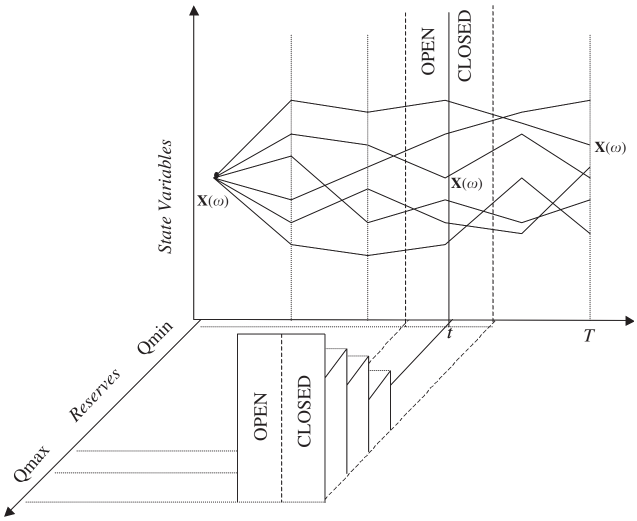

Fig. 3. State-space representation of the Brennan and Schwartz [23] model.

$\mathbf{x}(\omega) = [S(\omega), y(\omega), \upsilon(\omega)]$, evolve following different paths. At any point in time, and for any value of the three state variables, the mine may have any amount of copper reserves between zero and the initial reserves $Q_{\max}$. In addition, the mine at that point may be open or closed with market values $V_t(\mathbf{x}(\omega), Q)$ or $W_t(\mathbf{x}(\omega), Q)$,[3] respectively.

For each state of the system and for each operating policy, there is an associated cash flow for the mine. For example, when the mine is open and the operating policy is to remain open during Δt years producing q, the cash flow, CF, is

$$CF(S, q) = q\Delta t(S - A) - \tau. \tag{20}$$

Recall that for any price model, the spot price depends on the state variables $\mathbf{x}$, i.e. $S = f(\mathbf{x})$. In particular, for the three-factor Cortazar and Schwartz [7] model used in this paper, we have:

$$S = f(\mathbf{x}) = \mathbf{h}'\mathbf{x} \quad \text{with} \quad \mathbf{h}' = [1 \ \ 0 \ \ 0]. \tag{21}$$

Also, as noted previously, the mine may be open, closed or abandoned, and may switch from one operating state to another incurring in fixed costs.

Fig. 4 summarizes the cash flows of an open mine which will either remain open, be closed or abandoned during time t. Fig. 5 shows the same information, but for a closed mine.

[3] In Section 2.2 the status of the mine (open or closed) was indicated using the variable j.

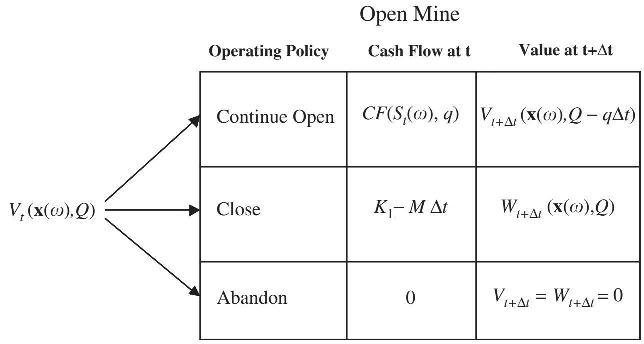

Fig. 4. Cash flows and value of an open mine as a function of the operating policy.

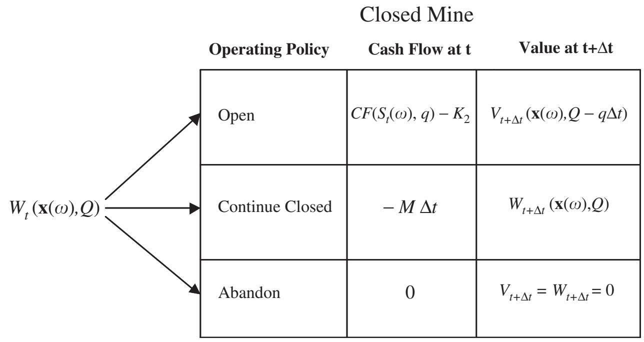

Fig. 5. Cash flows and value of a closed mine as a function of the operating policy.

As described earlier, after simulating all price paths from time zero to time T, the method requires making optimal decisions starting at time T and then working backwards until time zero is reached. The optimal decision at each point is taken by maximizing market value among all available alternatives.

At time T, given that the concession ends, the value of both the open and the closed mine is zero:

$$V_T(\mathbf{x}(\omega), Q) = W_T(\mathbf{x}(\omega), Q) = 0 \quad \forall Q, \ \forall \omega. \tag{22}$$

Then, at $t = T - \Delta t$ there is no time left to change the operating policy so there is no need to estimate an expected continuation value. So the market values are:

$$V_{T-\Delta t}(\mathbf{x}(\omega), Q) = \mathrm{Max}(CF(S_{T-\Delta t}(\omega), q); 0) \quad \forall Q, \tag{23}$$

$$W_{T-\Delta t}(\mathbf{x}(\omega), Q) = \mathrm{Max}(CF(S_{T-\Delta t}(\omega), q) - K_2; 0) \quad \forall Q. \tag{24}$$

Then, at $t = T - 2\Delta t$ we must estimate the expected continuation value. We regress the discounted mine value on a linear combination of functional forms of the state variables L(X), for each inventory level Q:

$$[V_{T-\Delta t}(\mathbf{X}, Q)\mathrm{e}^{-(r+\lambda_1)\Delta t} | W_{T-\Delta t}(\mathbf{X}, Q)\mathrm{e}^{-(r+\lambda_0)\Delta t}] = \mathbf{L}_{T-2\Delta t}(\mathbf{X})[a_{V,Q,T-2\Delta t} | a_{W,Q,T-2\Delta t}] + \mathrm{e}. \tag{25}$$

Once the optimal coefficients are found we can estimate the expected continuation values at $t = T - 2\Delta t$:

$$[\hat{\mathbf{G}}_{V,Q,T-2\Delta t} | \hat{\mathbf{G}}_{W,Q,T-2\Delta t}] = \mathbf{L}_{T-2\Delta t}(\mathbf{X})[\hat{a}_{V,Q,T-2\Delta t} | \hat{a}_{W,Q,T-2\Delta t}]. \tag{26}$$

Table 2
Expected and realized value of an open mine as a function of the operating policy

Expected value	Optimal decision	Realized value
$CF(S_t(\omega), q) + \hat{G}_{v,Q-q\Delta t,t}(\mathbf{x}(\omega))$	Continue open	$V_t(\mathbf{x}(\omega), Q) = CF(S_t(\omega), q) + V_{t+\Delta t}(\mathbf{x}(\omega), Q - q\Delta t)e^{-(r+\lambda_1)\Delta t}$
$-K_1 - M\Delta t + \hat{G}_{W,Q,t}(\mathbf{x}(\omega))$	Close	$V_t(\mathbf{x}(\omega), Q) = -K_1 - M\Delta t + W_{t+\Delta t}(\mathbf{x}(\omega), Q)e^{-(r+\lambda_0)\Delta t}$
0	Abandon	$V_t(\mathbf{x}(\omega), Q) = 0$

Table 3
Expected and realized value of a closed mine as a function of the operating policy

Expected value	Optimal decision	Realized value
$-K_2 + CF(S_t(\omega), q) + \hat{G}_{v,Q-q\Delta t,t}(\mathbf{x}(\omega))$	Open	$W_t(\mathbf{x}(\omega), Q) = -K_2 + CF(S_t(\omega), q) + V_{t+\Delta t}(\mathbf{x}(\omega), Q - q\Delta t)e^{-(r+\lambda_1)\Delta t}$
$-M\Delta t + \hat{G}_{W,Q,t}(\mathbf{x}(\omega))$	Continue closed	$W_t(\mathbf{x}(\omega), Q) = -M\Delta t + W_{t+\Delta t}(\mathbf{x}(\omega), Q)e^{-(r+\lambda_0)\Delta t}$
0	Abandon	$W_t(\mathbf{x}(\omega), Q) = 0$

Table 4
Open mine values as a function of the initial operation decision

Continue open	$V_0(\mathbf{x}, Q) = CF(S_0, q) + \hat{G}_{V,Q-q\Delta t,t=0}(\mathbf{x})$
Close	$V_0(\mathbf{x}, Q) = -K_1 - M\Delta t + \hat{G}_{W,Q,t=0}(\mathbf{x})$
Abandon	$V_0(\mathbf{x}, Q) = 0$

Thus, the expected continuation value at time $t = T - 2\Delta t$, as a function of the price state vector $\mathbf{x}$, may be computed. For example, the value of an open mine with Q units of resources, conditional on the state vector $\mathbf{x}$, would be

$$\hat{G}_{V,Q,T-2\Delta t}(\mathbf{x}) = \sum_{j=1}^{M} \hat{a}^j_{V,Q,T-2\Delta t} L^j_{T-2\Delta t}(\mathbf{x}). \tag{27}$$

Given that we can compute the expected continuation value, we are now able to obtain the optimal operating decisions by maximizing current cash flows plus the present value of expected continuation values. For example, when the mine is open there are three available operating alternatives: to continue open, to close down operations, or to abandon the mine. Adding current cash flows to discounted expected continuation values for each of the three alternatives, the decision maker may choose the best course of action.

Table 2 shows, for each of the three alternatives, the expected present value (at time t), the optimal decision should this expected present value be the maximum among the alternatives, and the final value at time t using actual realizations of the price simulation (instead of expected values to avoid biases due to the Jensen's inequality) at time $t + 1$. Table 3 shows the same information, but when the mine is initially closed.

This procedure is repeated from $t = T - 2\Delta t$ until $t = \Delta t$. At $t = \Delta t$ mine values are averaged over all price paths to provide an initial estimate of the expected continuation value for the mine:

$$\hat{G}_{V,Q-q\Delta t,t=0}(\mathbf{x}) = \frac{1}{s} \sum_{\omega=1}^{s} V_{\Delta t}(\mathbf{x}(\omega), Q - q\Delta t)\mathrm{e}^{-(r+\lambda_1)\Delta t}, \tag{28}$$

$$\hat{G}_{W,Q,t=0}(\mathbf{x}) = \frac{1}{s} \sum_{\omega=1}^{S} W_{\Delta t}(\mathbf{x}(\omega), Q)e^{-(r+\lambda_0)\Delta t}. \tag{29}$$

Tables 4 and 5 show the initial mine values depending on the initial status and operating policy of the mine.

Finally, to determine the optimal operating policy the method must find the critical state variables, $\mathbf{x}^c$, which equate expected present values for different operating decisions.

Table 5
Closed mine values as a function of the initial operation decision

Open	$W_0(\mathbf{x}, Q) = -K_2 + CF(S_0, q) + \hat{G}_{V,Q-q\Delta t,t=0}(\mathbf{x})$
Continue closed	$W_0(\mathbf{x}, Q) = -M\Delta t + \hat{G}_{W,Q,t=0}(\mathbf{x})$
Abandon	$W_0(\mathbf{x}, Q) = 0$

Table 6
Conditions to determine critical state variables $\mathbf{x}^c$ for switching mine operation

Open to Closed	$CF(\mathbf{x}^c, q) + \hat{G}_{V,Q-q\Delta t,t}(\mathbf{x}^c) = -K_1 - M\Delta t + \hat{G}_{W,Q,t}(\mathbf{x}^c)$
Closed to Open	$-M\Delta t + \hat{G}_{W,Q,t}(\mathbf{x}^c) = -K_2 + CF(\mathbf{x}^c, q) + \hat{G}_{V,Q-q\Delta t,t}(\mathbf{x}^c)$
Open to Abandon	$CF(\mathbf{x}^c, q) + \hat{G}_{V,Q-q\Delta t,t}(\mathbf{x}^c) = 0$
Closed to Abandon	$-M\Delta t + \hat{G}_{W,Q,t}(\mathbf{x}^c) = 0$

Table 7
Restrictions on initial state variables and parameters of the Cortazar and Schwartz [7] model to induce a one-factor price process similar to the Brennan and Schwartz [23] model

Cortazar–Schwartz model	Brennan–Schwartz model
y_o	λ_2/κ
v_0	$\overline{v} - \lambda_3/a$
λ_1	$v_0 - y_0 - (r - c)$
λ_2	≈ 0
λ_3	≈ 0
a	1
κ	1
$\overline{v}$	≈ 0
σ_1	σ
σ_2	≈ 0
σ_3	≈ 0
ρ_{12}	≈ 0
ρ_{23}	≈ 0
ρ_{13}	≈ 0

Table 6 shows how to find the critical state variables to close an open mine, to open a closed mine, or to abandon from an open or from a closed mine.

4. Results

4.1. Results for the one-factor Brennan and Schwartz [23] model

In this section we validate our proposed approach by applying it to the one-factor Brennan and Schwartz [23] real options model and comparing the results to those originally reported using traditional finite difference methods.

A simple way of validating our approach is to see the one-factor price process as a particular case of the more general three-factor process. In this way by restricting some parameter values we can perform a better test on the algorithm by using the same computer program to solve both models.

Table 7 shows how the Cortazar and Schwartz [7] three factor model may be restricted to behave as the one-factor model used in Brennan and Schwartz [23]:

The simulation program computed 50 000 price paths, assuming a maximum extraction time of 50 years with three opportunities per year to switch between operating states. This is an approximation to the continuous-time Brennan and Schwartz model which assumes an infinite concession time and infinite opportunities per year to switch operating states.

Table 8
Open and closed mine value as a function of spot price

Spot price (US$lb.)	Mine value finite difference method reported in [23]		Mine value Simulation method	
	Open	Closed	Open	Closed
0.4	4.15	4.35	4.2	4.4
0.5	7.95	8.11	7.93	8.12
0.6	12.52	12.49	12.51	12.49
0.7	17.56	17.38	17.51	17.31
0.8	22.88	22.68	22.8	22.6
0.9	28.38	28.18	28.29	28.09
1.0	34.01	33.81	33.89	33.69

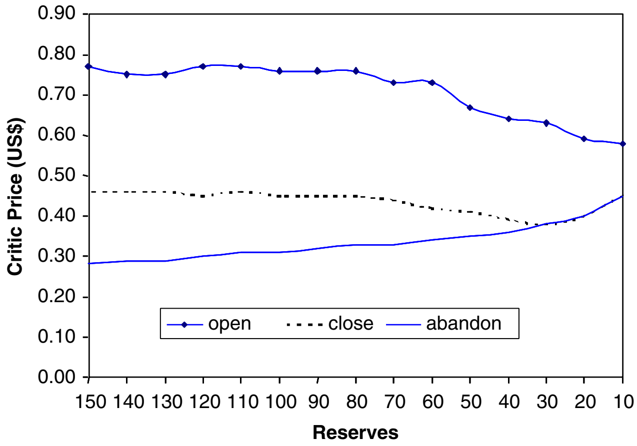

Fig. 6. Critical prices for opening, closing or abandoning a mine, as a function of reserve level obtained using the LSM method.

Table 8 compares the finite difference values reported in [23] with those obtained using the above simulation procedure. The mine and market parameters used are those reported in [23]. It can be seen that the simulation method converges to the known finite difference solution.

Our simulation procedure may also provide the optimal operating policy. Fig. 6 shows the critical prices for abandoning, opening a closed mine, and closing an open mine, as a function of reserves. Results are very similar to those reported in [23].

4.2. *Results for the three-factor extension of the Brennan and Schwartz [23] model*

We now report the solution to the Brennan and Schwartz [23] model extended to include the Cortazar and Schwartz [7] three-factor commodity price model. The parameter values used are those reported in Table 1.

We now assume a 30 year concession horizon, and three opportunities to switch operation states per year. To value the mine for a particular date, say April the 14th, 1999, we must first determine the values of the state variables S_o, y_o, υ_o corresponding to that date, which are 0.64, 0.198 and 0.244, respectively. Following the implementation procedure described in Section 4.1 we obtain a value for the open mine of MMUS$ 15.64, and for the closed mine of MMUS$ 15.52.

To explore how mine value changes according to variations in price conditions, we solve for the value of the mine for a 5 year time span. Results are reported in Fig. 7.

It is interesting to note that mine value exhibits mean reversion. Even though it is well known that copper prices do exhibit mean reversion, which is captured in the three-factor model, given that a mine produces copper during a long

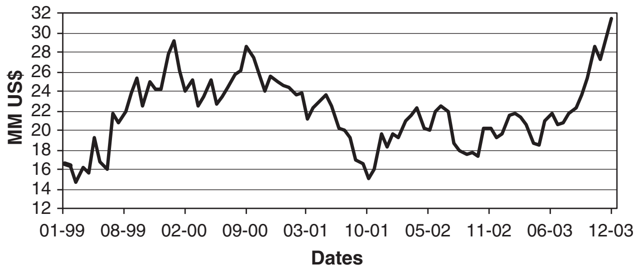

Fig. 7. Monthly values of the extended Brennan and Schwartz [23] open mine according to historical copper pricing conditions from January 1999–December 2003.

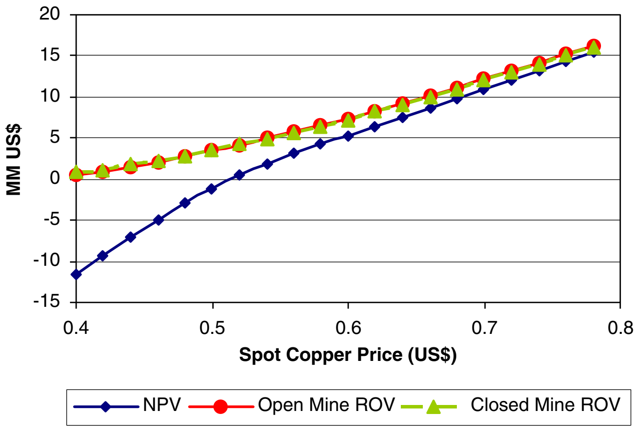

Fig. 8. Value of the open mine using ROV and NPV as a function of spot price for $y = 0.01$ and $v = -0.1$.

time horizon it could be thought that current spot prices would not have a great effect on mine values. Fig. 7 shows this is not the case.

Doing comparative static analysis on how mine value changes with variations in the spot price or in any individual state variable or parameter value is rather straightforward. For example, Fig. 8 shows how mine value increases with copper spot prices. It is also interesting to note how mine values are convex, because as mine value approaches zero the probability of abandoning the mine increases. Finally, the same figure compares mine value computed with the real option model to a simple net present value calculation which does not recognize operating flexibilities to abandon or close operations. It can be seen that when spot prices are lower, option values are greater and these two valuation methodologies diverge the most. By the same token, when prices are high, flexibilities are not too valuable and both valuations converge.

Comparative static analysis for the value or for the optimal policy can easily be performed for any of the state variables, strengthening the ability of the LSM method to study the behavior of an investment project for different scenarios.

4.3. An alternative implementation for multi-dimensional settings

In the previous sections we have shown a simple implementation of the LSM approach for solving a real options model with a three-factor price process. As stated previously, one of the main contributions of this approach is the computation of the expected continuation value by regressing discounted future option values on a linear combination of functional forms of current state variables. The way these functional forms are chosen is not straightforward and, as is discussed in this section, it may become an important issue in high-dimensional settings.

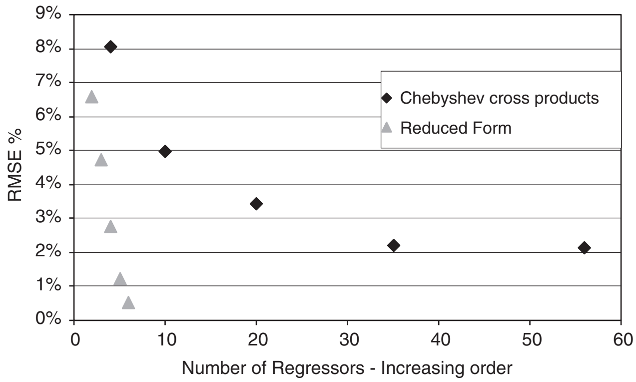

Fig. 9. RMSE as a function of the number of regressors for Chebyshev Polynomials and for the reduced-base form using only futures.

Longstaff and Schwartz [19] propose for multidimensional implementations of their method the use of basic functions from Laguerre, Chebyshev, Gegenbauer, Jacobi polynomials, or, the simple powers and cross products of the state variables used in this paper. For example, if the state variables were only two, X and Y, a simple order-two expected continuation value function would have six regressors, namely:

$$\hat{G}(X,Y) = \hat{a}_0 + \hat{a}_1 X + \hat{a}_2 Y + \hat{a}_3 XY + \hat{a}_4 X^2 + \hat{a}_5 Y^2. \tag{30}$$

Although this procedure for specifying the regression basis has the benefit of being simple and theoretically convergent [22,52,53], in high-dimensional settings it may induce numerical problems due to the least squares regression instability [21] and performance problems due to the high number of regressors.

An alternative to the described procedure for specifying the base that we have tested is to take advantage of the structure of the problem to be solved. Thus, given that optimal exercise of options depends on expected spot prices and volatilities, instead of using as regressors powers of all state variables, it could be better to use functions on futures, European options or bond prices, which have economic meaning.

Recent independent work has shown the potential of this approach for implementing multidimensional financial derivatives. For example Andersen and Broadie [50] include as regressors European call options and their powers for valuing a multi-stock option and Longstaff [51] value the prepayment option on a term structure string model with 120 state variables using closed form par-price bonds and their powers. We are not aware, however, of any use of a similar approach in the real options literature.

Thus our alternative implementation, in its simplest specification, boils down to computing the expected continuation value function:

$$\hat{G}_N(\mathbf{x}) = \hat{a}_0 + \sum_{i=1}^{N} \hat{a}_i E(S)^i, \tag{31}$$

where $E(S)$ is the expected spot price under the risk-adjusted measure, i.e., the future price.

Our tests show that using this reduced-base specification we can obtain similar valuation accuracy in a simpler way than using polynomials of state variables. For example, we solved a three-factor European option with known analytic solution with two alternative implementations of the LSM approach: Chebyshev functions and futures prices. Fig. 9 computes the RMSE as a function of the number of regressors, showing that using futures requires less regressors for any giver error level.

Using less regressors for estimating the continuation function has many computational benefits including reducing CPU-processing time which could be critical for high-dimensional implementations.

For example we performed another test solving the extended three-factor price model Brennan and Schwartz mine, obtaining valuations within 1% for both LSM implementations, while calculation time increased with the number of regressors, as shown in Fig. 10. These results suggest that if calculation time is an issue it is worth exploring alternative implementations of the LSM approach.

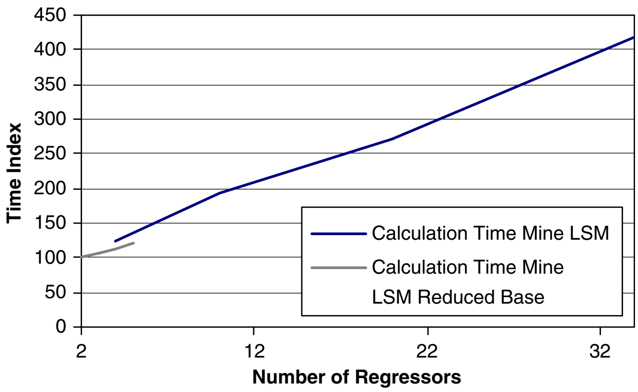

Fig. 10. Relative computer calculation time for solving the extended Brennan and Schwartz mine model as a function of the number of regressors when using the standard and the reduced base implementation of the LSM method.

5. Conclusions

Real options valuation (ROV) is an emerging paradigm that provides helpful insights for both valuing and managing real assets. It provides more precise quantifications on the value of available strategic and operational flexibilities than traditional discounted cash flow techniques.

Despite its potential, the ROV approach has not yet made a strong inroad in corporate decision-making due to several reasons, one of which is the requirement to keep models too simple to obtain solutions within a reasonable amount of effort.

In this paper we show how it is possible to solve complex multidimensional American options using computer-based simulation procedures. The implementation is validated using the one-factor Brennan and Schwartz [23] model with the reported finite difference solution.

We then extend the Brennan and Schwartz [23] to include a three-factor price model and solve it using the proposed methodology. Comparative static analyses are provided.

This paper argues that these new simulation methods have the potential of expanding significantly the use of the ROV approach without having to compromise rigorous modeling in order to obtain a solution.

Acknowledgments

We thank professor Eduardo Schwartz (UCLA), the researchers of the FINlabUC- Laboratorio de Investigación Avanzada en Finanzas- Pontificia Universidad Católica de Chile- and the participants of the 9th Annual International Conference of Real Options, Paris, June 2005, for helpful discussions. We also thank the financial support of FINlabUC, FONDECYT (Grant 1040608), FONDEF (Grant D03I1039) and FUNDACION COPEC-UNIVERSIDAD CATOLICA (Grant PC00021).

References

[1] Black F, Scholes M. The pricing of options and corporate liabilities. Journal of Political Economy 1973;81:637–54.
[2] Gibson R, Schwartz ES. Stochastic convenience yield and the pricing of oil contingent claims. The Journal of Finance 1990;45(3):959–76.
[3] Schwartz ES. The stochastic behavior of commodity prices: implications for valuation and hedging. The Journal of Finance 1997;52(3): 923–73.
[4] Schwartz ES, Smith JE. Short-term variations and long-term dynamics in commodity prices. Management Science 2000;46:893–911.
[5] Casassus J, Collin-Dufresne P. Stochastic convenience yield implied from commodity futures and interest rates. Journal of Finance 2005;60(5):2283–331.
[6] Sørensen C. Modeling seasonality in agricultural commodity futures. Journal of Futures Markets 2002;22:393–426.
[7] Cortazar G, Schwartz ES. Implementing a stochastic model for oil futures prices. Energy Economics 2003;25(3):215–38.
[8] Hsu J, Schwartz ES. A Model of R&D Valuation and the Design of Research Incentives. UCLA: Anderson School; 2003.

[9] Cortazar G, Schwartz E, Casassus J. Optimal exploration investments under price and geological-technical uncertainty: a real options model. R&D Management 2001;31(2):181–9.
[10] Kulatilaka N. Operating flexibilities in capital budgeting: substitutability and complementarity in real options. In: Trigeorgis L, editor. Real options in capital investments: new contributions. New York: Praeger, 1995.
[11] Bernardo A, Chowdry B. Resources, real options and corporate strategy. Journal of Finance 2002;63:211–34.
[12] Bossaerts P. Simulation estimators of early optimal exercise. Graduate School of Industrial Administration, Carnegie Melon University; 1988.
[13] Tilley JA. Valuing American options in a path simulation model. Transactions of the Society of Actuaries 1993;45:42–56.
[14] Barraquand J, Martineau D. Numerical valuation of high dimensional multivariate American securities. Journal of Financial and Quantitative Analysis 1995;30(3):301–20.
[15] Raymar S, Zwecher M. A Monte Carlo valuation of American call options on the maximum of several stocks. Journal of Derivatives 1997;5(1, (Fall)):7–23.
[16] Broadie M, Glasserman P. Pricing American-style securities using simulation. Journal of Economics Dynamics and Control 1997;21(8): 1323–52.
[17] Andersen L. A simple approach to the pricing of Bermudian swaptions in the multi-factor libor market model. Journal of Computational Finance 2000;3:5–32.
[18] Haugh M, Kogan L. Approximating pricing and exercising of high-dimensional American options: a duality approach. Cambridge, MA: MIT; 2001.
[19] Longstaff FA, Schwartz ES. Valuing American options by simulation: a simple least-squares approach. The Review of Financial Studies 2001;14(1):113–47.
[20] Stentoft L. Assessing the least squares Monte-Carlo approach to American option valuation. Review of Derivatives Research 2004;7(2): 129–68.
[21] Moreno M, Navas J. Review of Derivatives Research 2003;6(2):107–28.
[22] Clement E, Lamberton D, Protter P. An analysis of a least squares regression method for American option pricing. Finance and Stochastics 2002;6(4):449–71.
[23] Brennan MJ, Schwartz ES. Evaluating natural resources investments. Journal of Business 1985;58(2):135–57.
[24] Majd S, Pindyck R. The learning curve and optimal production under uncertainty. Rand Journal of Economics 1989;29:1110–48.
[25] Trigeorgis L. The nature of option interactions and the valuation of investments with multiple real options, Journal of Financial and Quantitative Analysis 1993; 1–20.
[26] McDonald R, Siegel D. The value of waiting to invest. Quarterly Journal of Economics 1986;101:707–27.
[27] Majd S, Pindyck R. Time to build, option value, and investment decisions. Journal of Financial Economics 1987;18(2):7–27.
[28] He H, Pindyck R. Investment in flexible production capacity. Journal of Economics Dynamics and Control 1992;16:575–99.
[29] Cortazar G, Schwartz ES. A compound option model of production and intermediate investment. Journal of Business 1993;66(4):517–40.
[30] Dixit A. Entry and exit decisions under uncertainty. Journal of Political Economy 1989;97:620–38.
[31] Pindyck R. Investment of uncertain cost. Journal of Financial Economics 1993;34(1):53–76.
[32] Ekern S. An option pricing approach to evaluating petroleum projects. Energy Economics 1988;10:91–9.
[33] Trigeorgis L. Evaluating leases with complex operating options. European Journal of Operations Research 1996;91:69–86.
[34] Brennan MJ, Trigeorgis L. Project flexibility, agency, and competition. Oxford: Oxford University Press; 2000.
[35] Dixit A, Pindyck R. Investment under uncertainty. Princeton, NJ: Princeton University Press; 1994.
[36] Decamps J, Mariotti T, Villeneuve S. Investment timing under incomplete information, European Economic Association Annual Congress, 2003.
[37] Lambrecht B, Perraudin W. Real options and preemption under incomplete information. Journal of Economic Dynamics and Control 2003;27(4):619–43.
[38] Miltersen KR, Schwartz ES. R&D investments with competitive interactions. NBER: Cambridge, MA; 2003.
[39] Murto P, Nasakkala E, Keppo J. Timing of investments in oligopoly under uncertainty: a framework for numerical analysis. European Journal of Operations Research 2004;157(2):486–500.
[40] Paddock J, Siegel D, Smith J. Option valuation of claims on physical assets: the case of offshore petroleum leases. Quarterly Journal of Economics 1988;103(3):479–508.
[41] Cortazar G, Casassus J. Optimal timing of a mine expansion: implementing a real options model. The Quarterly Review of Economics and Finance 1998;38:755–69.
[42] Smith J, McCardle K. Valuing oil properties: integrating option pricing and decision analysis approaches. Operations Research 1998;46: 198–217.
[43] Smith J, McCardle K. Options in the real world: lessons learned in evaluating oil and gas investments. Operations Research 1999;47:1–15.
[44] Lehman J. Valuing oilfield investments using option pricing theory. SPE Hydrocarbon Economics and Evaluation Symposium. Proceedings 1989. p. 125–36.
[45] Trigeorgis L. A real options application in natural resource investment. Advances in Futures and Options Research 1990;4:153–64.
[46] Laughton DG, Jacoby HD. Reversion, timing options, and long-term decision-making. Financial Management 1993;22(3):225–40.
[47] Laughton DG, Jacoby HD. The effects of reversion on commodity projects of different length. In: Trigeorgis L, editor. Real options in capital investments: models, strategies, and applications. Westport: Praeger Publisher; 1995.
[48] Cortazar G, Schwartz ES. Implementing a real option model for valuing an undeveloped oil field. International Transactions in Operational Research 1997;4(2):125–37.
[49] Cortazar G, Naranjo L. An *N*-factor Gaussian model of oil futures prices. Journal of Futures Markets 2006;26(3):243–68.
[50] Andersen L, Broadie M. A primal-dual simulation algorithm for pricing multi-dimensional American options. Management Science 2004;50(9):1222–34.

[51] Longstaff, F. Optimal recursive refinancing and the valuation of mortgage-backed securities, NBER Working Papers 10422, 2004.
[52] Glasserman P, Yu B. Number of paths versus number of basis functions in American option pricing. Annals of Applied Probability 2004;14(4):2090–119.
[53] Stentoft L. Convergence of the least squares Monte Carlo approach to American option valuation. Management Science 2004;50(9):1193–203.